Forum Geometricorum
Volume 18 (2018) 17–36.

FORUM GEOM
ISSN 1534-1178

Areas and Shapes of Planar Irregular Polygons

C. E. Garza-Hume, Maricarmen C. Jorge, and Arturo Olvera

Abstract. We study analytically and numerically the possible shapes and areas of planar irregular polygons with prescribed side-lengths. We give an algorithm and a computer program to construct the cyclic configuration with its circumcircle and hence the maximum possible area. We study quadrilaterals with a self-intersection and prove that not all area minimizers are cyclic. We classify quadrilaterals into four classes related to the possibility of reversing orientation by deforming continuously. We study the possible shapes of polygons with prescribed side-lengths and prescribed area.

In this paper we carry out an analytical and numerical study of the possible shapes and areas of general planar irregular polygons with prescribed side-lengths. We explain a way to construct the shape with maximum area, which is known to be the cyclic configuration. We write a transcendental equation whose root is the radius of the circumcircle and give an algorithm to compute the root. We provide an algorithm and a corresponding computer program that actually computes the circumcircle and draws the shape with maximum area, which can then be deformed as needed. We study quadrilaterals with a self-intersection, which are the ones that achieve minimum area and we prove that area minimizers are not necessarily cyclic, as mentioned in the literature ([4]).

We also study the possible shapes of polygons with prescribed side-lengths and prescribed area. For areas between the minimum and the maximum, there are two possible configurations for quadrilaterals and an infinite number for polygons with more than four sides.

The work was motivated by the study of two Codices, ancient documents from central Mexico written around 1540 by a group called the Acolhua ([9],[10] or online [3]). The codices contain drawings of polygonal fields, with their side-lengths in one section and their areas in another. We had to find out if the proposed areas were correct and to confirm the location of some of the fields.

The fields are not drawn to scale and it is not known how the Acolhua computed (or measured) areas. There are no angles or diagonals therefore we could not compute the actual areas but one thing we could do was compute maximum and minimum possible areas for the given side-lengths and say the areas in the documents were feasible if they were between those two values. Maxima are easy to

Publication Date: January 17, 2018. Communicating Editor: Paul Yiu.

We thank Ana Pérez Arteaga and Ramiro Chávez Tovar for computational support, figures and animations. This work was supported by Conacyt Project 133036-F..

find for quadrilaterals by applying Brahmagupta's formula ([5], [6]) but we could not find in the literature a formula for the maximum possible area of polygons with more than four sides. There are some results in [8] but not a formula. We could also not find a complete treatment of minima in the literature.

The study of the documents also required an analysis of the possible shapes of polygons with prescribed side-lengths and area to try to localize the fields. For polygons with more than four sides there is no easily available, explicit formula for this. We explain the mathematics of the problem, provide an algorithm to find the possible shapes and a computer implementation in javascript of the algorithm.

Observe that finding areas of polygons is just an application of calculus *if* the coordinates of the vertices are known but the problem we are addressing here is, what can be done when only side-lengths are known, not angles, diagonals or coordinates. The question is relevant in surveying, architecture, home decorating, gardening and carpentry to name but a few.

For some quadrilaterals it is possible to change orientation by deforming continuously and in others it is necessary to perform a reflection. This observation led to a classification of quadrilaterals into four classes.

The case of triangles is special because they are the only rigid polygons, that is, the shape and area are determined by the side-lengths. Also, all triangles can be inscribed in a circle whose center is located at the intersection of the perpendicular bisectors. If the side-lengths are called a, b, c the area is given by Heron's Formula,

$$\mathrm{A}_H(a,b,c) = \sqrt{s(s-a)(s-b)(s-c)} \tag{1}$$

where $s = \frac{1}{2}(a+b+c)$ is the semi-perimeter. In what follows we will discuss polygons with more than three sides.

1. Computing the shape with maximum area for any polygon

In this section we will describe how to compute the maximum possible area of a planar polygon with n sides and how to construct the shape with maximum area given its ordered side-lengths. It is well-known that no side-length can be larger than the sum of the others if they are to form a closed polygon. We will call this the compatibility condition. It was proved in [4] that the configuration with maximum possible area is given by the cyclic polygon, that is, the configuration that can be inscribed in a circle. The problem is that we do not know *which* circle; we do not know its center or its radius. We will solve this problem in the present section.

We will first give an analytical solution for quadrilaterals and then describe an iterative method that works for any polygon and its numerical implementation.

Analytic solution for n=4: Circle-Line construction. For $n = 4$, Bretschneider's formula gives the (unoriented) area $\mathcal{A}$ of a quadrilateral with side-lengths a, b, c, d and opposite angles α and γ as

$$\mathcal{A} = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2(\frac{\alpha+\gamma}{2})}, \tag{2}$$

where $s = (a+b+c+d)/2$, see Figure 1. Observe that $\mathcal{A}$ is always non-negative.

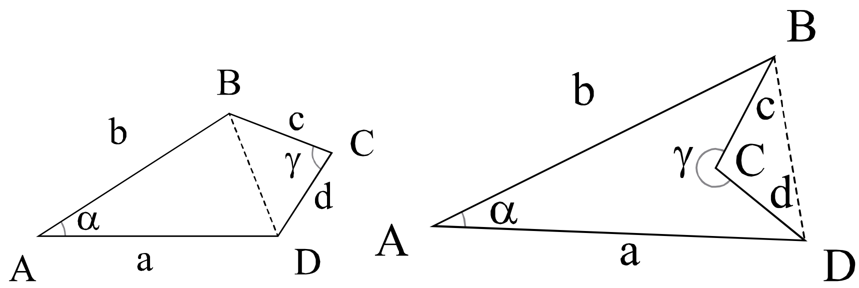

Figure 1. Two examples of a quadrilateral $ABCD$.

The maximum value of this expression is

$$\mathcal{A}_{max} = \sqrt{(s-a)(s-b)(s-c)(s-d)}, \tag{3}$$

and it is attained when the sum of either pair of opposite angles is π radians, that is, when the quadrilateral es cyclic. But this does not give directly the coordinates of the shape with maximum area.

There is a formula for the vertices of the shape with maximum area which is described in [6] SI Appendix p.414, for a quadrilateral with given side-lengths a, b, c, d and prescribed area $\mathcal{A}_c$. We give here the geometric idea.

As explained in [6], vertex A is placed at the origin and vertex D at $(a, 0)$; it remains to find the coordinates of vertices B and C. The coordinates of vertex B are (u, v) and of vertex C, (w, z) (see Figure 2).

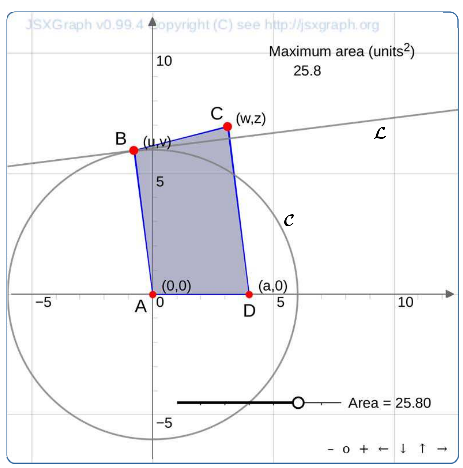

Figure 2. Shape with maximum area of a quadrilateral with side-lengths 4,6,4,7 arbitrary units.

After some lengthy computations with analytic geometry which we include in Appendix A, it is shown that vertex B is given by the intersection of a circle $\mathcal{C}$ of radius b and center at the origin, and a line $\mathcal{L}$ which depends on the side-lengths and on the prescribed area $\mathcal{A}_c$. The maximum area is attained when the line is tangent to the circle. The coordinates of vertex B are given explicitly and the coordinates of vertex C and found from the lengths of the remaining sides.

Figure 2 shows an example of maximum area produced using a program we wrote using routines from JSXGraph which can be found under “Circle-line construction” in www.fenomec.unam.mx/ipolygons. The value of the maximum area can be computed from equation (3). In this example, in arbitrary units $a = 4$, $b = 6$, $c = 4$, $d = 7$ therefore $s = \frac{1}{2}(4+6+4+7) = 21/2$ and the maximum possible area is approximately 25.8 square units. The program allows the computation of the *shape* with maximum area.

The construction also shows that if $\mathcal{A}_c > \mathcal{A}_{max}$ then the line and circle do not intersect and hence no quadrilateral exists with those side-lengths and area. If $\mathcal{A}_{min} < \mathcal{A}_c < \mathcal{A}_{max}$ then $\mathcal{L}$ intersects $\mathcal{C}$ twice giving two possible shapes for the given quadrilateral data. Here $\mathcal{A}_{min}$ represents the minimum possible area. We will say more about it later.

We recommend playing with the computer program to understand the construction and its implications.

Iterative method for any $n \geq 3$ and its numerical implementation. For $n > 4$ there is no explicit formula for the center or radius r of the circumcircle of the configuration with maximum area.

We describe here an iterative method that works for any $n \geq 3$ for finding the circumcircle and thus the maximum area of the polygon and the coordinates of its vertices.

Observe first that the center of the circumcircle of a cyclic polygon can be inside, outside or on the boundary of the polygon, as illustrated in Figure 3, and this determines how the circumcircle is computed.

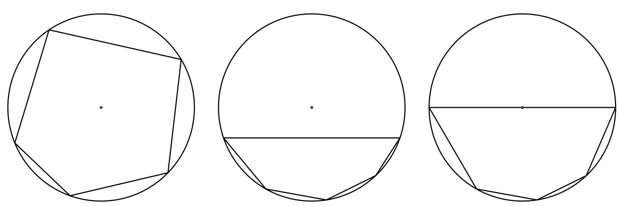

Figure 3. Position of the center of the circumcircle.

Now consider n side-lengths $a_1, a_2, \ldots, a_n$ that satisfy the compatibility condition, placed in clockwise order in the shape of maximum area.

Referring to Figure 4, by the cosine rule

$$a_j^2 = 2r^2 - 2r^2 \cos\theta_j$$

therefore

$$\cos\theta_j = \frac{2r^2 - a_j^2}{2r^2}.$$

It is not known *a priori* if the center of the circumcircle will be inside or outside the polygon. If it is inside the polygon (Figure 4, left), then $\sum_{j=1}^{n} \theta_j = 2\pi$ and

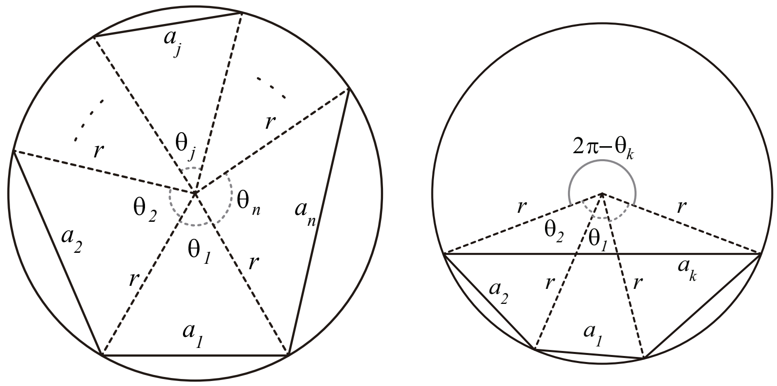

Figure 4. Determination of r when the center of the circumcircle is inside (left) and outside (right) the polygon.

therefore

$$\sum_{j=1}^{n} \text{Arccos}(1 - \frac{a_j^2}{2r^2}) = 2\pi.$$

From this observation we define a function $F_1(r)$ whose root is the radius of the circumcircle when the center is inside the polygon:

$$F_1(r) = \sum_{j=1}^{n} \text{Arccos}(1 - \frac{a_j^2}{2r^2}) - 2\pi = 0. \tag{4}$$

We have used the normal convention that Arccos refers to the principal branch and arccos refers to the secondary branch which is given by $2\pi -$ Arccos.

If the center of the circumcircle is outside the polygon (Figure 4, right) the angle subtended at the center by the longest side is equal to the sum of the angles subtended by the other sides, or equivalently, the inverse cosine of the longest side has to be taken in the secondary branch. We define another function, $F_2(r)$ whose root is the radius of the circumcircle when the center is outside the polygon. If we call a_k the longest side,

$$\begin{aligned} F_2(r) &= \sum_{j \neq k}^{n} \text{Arccos}(1 - \frac{a_j^2}{2r^2}) + \text{arccos}(1 - \frac{a_k^2}{2r^2}) - 2\pi \\ &= \sum_{j \neq k}^{n} \text{Arccos}(1 - \frac{a_j^2}{2r^2}) + 2\pi - \text{Arccos}(1 - \frac{a_k^2}{2r^2}) - 2\pi \\ &= \sum_{j \neq k}^{n} \text{Arccos}(1 - \frac{a_j^2}{2r^2}) - \text{Arccos}(1 - \frac{a_k^2}{2r^2}) \\ &= 0. \end{aligned} \tag{5}$$

The radius of the circumcircle, r_M, is the root of $F_1(r)$ in the first case and of $F_2(r)$ in the second. If the center of the circumcircle is on the boundary of the polygon then both expressions coincide. One would normally look for the root numerically using a Newton method. However, in this case the graphs of F_1 and F_2 are almost vertical at the root and a Newton method is not adequate. A bisection method or quasi-Newton will work as long as one can find an interval $[r_l, r_r]$ that contains the root, that is, two values, r_l, r_r such that the function has opposite signs at r_l and r_r. The value r_l is easy to find because the inverse cosine is only defined between -1 and +1. Therefore, for all j,

$$-1 \leq 1 - \frac{a_j^2}{2r^2}$$

and so $r \geq a_j/2$ and the minimum positive value of r for which F_1 and F_2 are defined is $r_l = \max_j(a_j/2)$. Finding r_r is not so easy because when one side of the polygon approaches the sum of the others, the radius of the circumcircle tends to infinity.

The graphs of F_1 from equation (4) and F_2 from equation (5) for side-lengths 6,7,8,9,10 units are shown in Figure 5 and for side-lengths 9,9,9,9,29 in Figure 6.

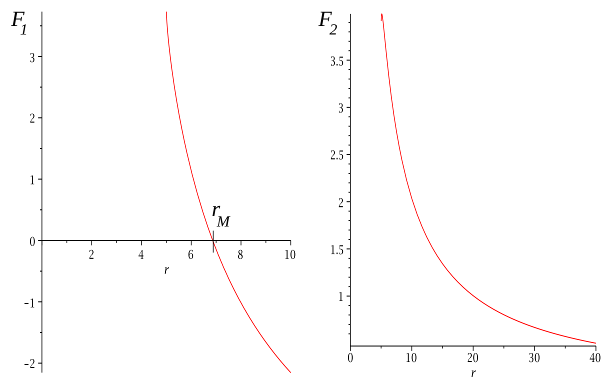

Figure 5. Graph of F_1 (left) and F_2 (right) for side-lengths 6,7,8,9,10. Only F_1 has a root, which is r_M, the radius of the circumcircle.

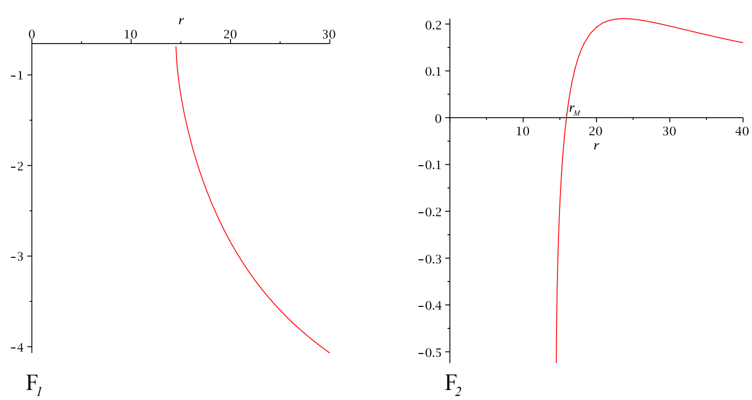

Figure 6. Graph of F_1 (left) and F_2 (right) for side-lengths 9,9,9,9,29. Only F_2 has a root, r_M, the radius of the circumcircle.

If F_1 has a root then the center is inside the polygon and F_2 does not have a root. Otherwise, F_2 has a root and the center is outside the polygon (see the proof in Appendix B.)

Once the radius r_M of the circumcircle $\mathcal{C}_{r_M}$ is calculated, then the area of the polygon with maximum area can be computed for instance by triangulating as in Figure 4 . The area of each triangle can be computed from Heron's formula and we obtain, if r_M was computed from F_1,

$$A = \sum_{i=1}^{n} \mathrm{A}_H(a_i, r_M, r_M)$$

and if r_M was computed from F_2,

$$A = \sum_{\substack{i=1\\ i\neq k}}^{n} \mathrm{A}_H(a_i, r_M, r_M) - \mathrm{A}_H(a_k, r_M, r_M).$$

To draw the polygon with maximum area first draw $\mathcal{C}_{r_M}$ with center at the origin. Then it is easiest to start with the longest side, a_k, place its right end at the point $P = (r_M, 0)$ and mark the two intersections of $\mathcal{C}_{r_M}$ with the circle of center P and radius a_k. Call i_1 the intersection in the lower semicircle and i_2 the one in the upper semicircle (see Figure 7.) If $F_1(r_M) = 0$ then the left end of side a_k is placed at i_1 and if $F_2(r_M) = 0$ then it is placed at i_2. The remaining vertices can be found by marking on $\mathcal{C}_{r_M}$ the side-lengths in clock-wise order. The coordinates of all the vertices can be calculated from this construction since they are intersections of $\mathcal{C}_{r_M}$ with circles of radius a_j and center at the previous vertex. The figure can then be rotated or translated as needed.

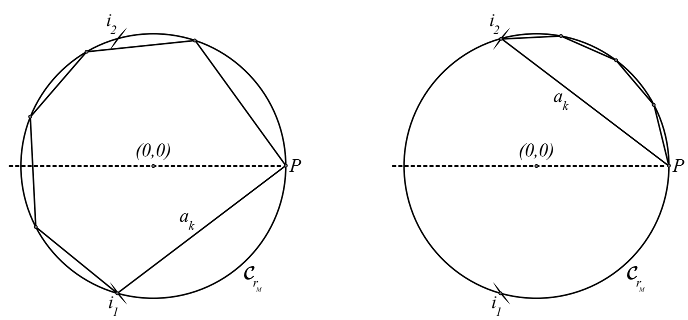

Figure 7. Generic construction of the shape with maximum area when F_1 has root (left) and when F_2 has root (right).

An implementation of this using javascript can be found in www.fenomec.unam.mx/ipolygons under "Build your own polygon".

Figure 8 shows the shapes corresponding to figures 5 and 6.

2. Note on oriented area

Now that we have the coordinates of the vertices, we could choose to compute the maximum area from the Shoelace formula ([7]), which is just an application

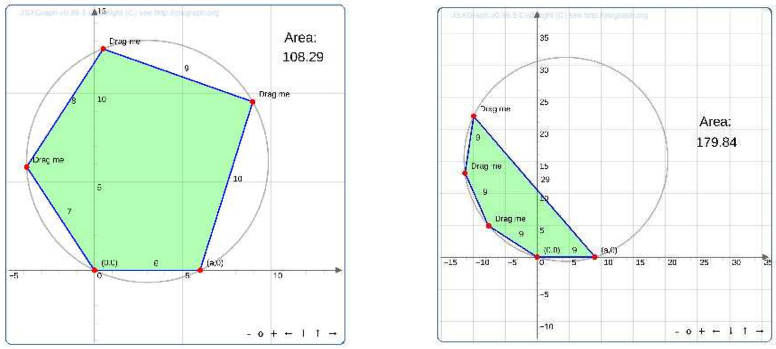

Figure 8. Shapes with maximum area for pentagon with sides 6,7,8,9,10 (left) and 9,9,9,9,29 (right).

of Green's Theorem. We already have the area from triangulation and Heron's formula but that is an unoriented area, it does not take into account the orientation of the sides. The Shoelace Formula does take orientation into account. For a polygon with vertices $A_0 = (x_0, y_0)$, $A_1 = (x_1, y_1)$, …, $A_n = (x_n, y_n)$ this formula can be written as

$$\text{Area} = \frac{1}{2}[(x_0y_1 + x_1y_2 + x_2y_3 + \cdots + x_ny_0) - (x_1y_0 + x_2y_1 + x_3y_2 + \cdots + x_0y_n)] \tag{6}$$

which is more easily remembered when written as in [7], shown in Figure (9).

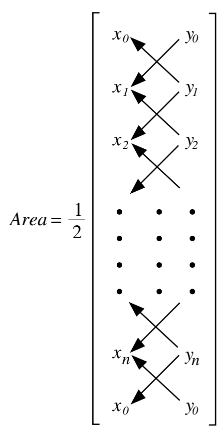

Figure 9. Shoelace formula

Arrows going up represent the products in the first bracket and arrows going down the products in the second bracket of (6). This gives a positive area if sides are taken in the positive direction, that is, counter-clockwise and negative otherwise.

We have actually used clockwise orientation as the positive one to be consistent with the data bases of the codices we created before.

The concept of oriented area is needed in the following sections.

3. Minima of irregular quadrilaterals

A quadrilateral is feasible if its prescribed area is between the maximum and the minimum possible values. To study minima of quadrilaterals one has to decide first whether to accept figures with self-intersections. If not, then the minimum will be attained by the smallest triangle into which the quadrilateral degenerates as the angles are moved (see Figure 10 for an example). This has been studied in [2] for any polygon. If self-intersections are admitted then one has to include the notion of signed area which was mentioned in the previous section.

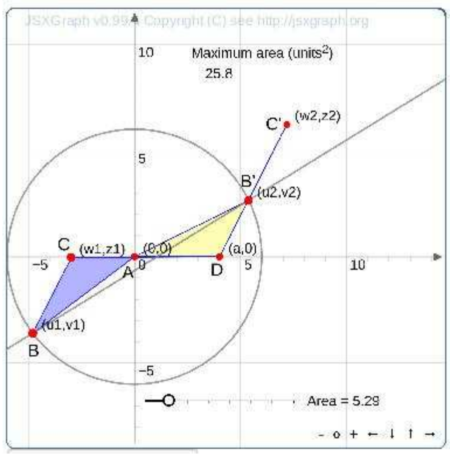

Figure 10. Two possible minima without self-intersection, $ABCD$ and $AB'C'D$, for quadrilateral with side-lengths 4,6,4,7. The side-lengths are the same as in Figure 2.

Decomposing the quadrilaterals in Figure 1 into two triangles we obtain

$$\mathcal{A} = \frac{1}{2}ab\sin\alpha + \frac{1}{2}cd\sin\gamma$$

which gives area $\triangle ABD +$ area $\triangle BCD$ if angle γ is less than π radians and area $\triangle ABD -$ area $\triangle BCD$ if γ is more than π radians because sine is then negative.

For quadrilaterals with self-intersections (see Figure 11) Bretschneider's Formula (equation (1)) yields the absolute value of the difference of the areas of the triangles ABE and EDC because triangle BED cancels out and because the formula computes the square root of the square of the area, which is the absolute value. We call this figure a quadrilateral with self-intersection or bow-tie because it arises by deforming a convex quadrilateral and its area is the *difference* of the areas of the two parts.

The minimum value of (2) is

$$\mathcal{A}B_{min} = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd} \tag{7}$$

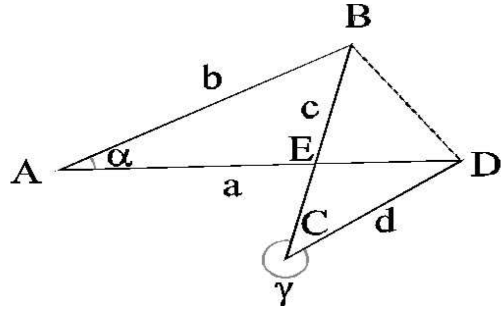

Figure 11. Quadrilateral with self-intersection or bow-tie ABCD.

but not all figures can achieve this theoretical minimum. If the expression inside the radical is negative, then $\mathcal{A}B_{min}$ is not defined over the reals and the minimum area, $\mathcal{A}_{min}$, is zero. When $\mathcal{A}B_{min}$ *is* achieved, $\alpha+\gamma=2\pi$ hence $\cos^2(\frac{\alpha+\gamma}{2})=1$ and the self-intersecting quadrilateral is cyclic. See appendix C for the proof. In reference [4] it is said that minima are cyclic but this is not always the case. It is only true when $(s-a)(s-b)(s-c)(s-d)-abcd \geq 0$. If $(s-a)(s-b)(s-c)(s-d)-abcd\cos^2(\frac{\alpha+\gamma}{2})$ becomes zero for some $\alpha+\gamma<2\pi$ then that configuration achieves the minimum area and it is NOT cyclic.

Minimum areas of quadrilaterals can also be studied using the construction given in [6]. A quadrilateral with given side-lengths and prescribed area $\mathcal{A}_c$ is obtained from the intersection of the circle and line from section 1. Let us analyze what happens when A_c goes from ∞ to 0 for a quadrilateral with fixed side-lengths. The slope of the line is $-P/Q$ where $Q=4a\mathcal{A}_c$ hence the line is horizontal for infinite area and gradually becomes more inclined until it becomes vertical at area $\mathcal{A}_c=0$, except when $P=0$, in which case the line is always horizontal[1]. If the sides do not satisfy the compatibility condition and hence no quadrilateral exists then the line never intersects the circle. Otherwise, the line becomes tangent, giving the configuration with maximum area, then intersects the circle twice, giving the two possible shapes of the quadrilateral with the prescribed area and then two things can happen: the line can become vertical before leaving the circle in which case $\mathcal{A}B_{min}$ is not achieved and the minima are not cyclic (see Figure 12) or the line can again become tangent to the circle, leave, and then become vertical. This other point of tangency gives the cyclic configuration with area $\mathcal{A}B_{min}$ (see Figure 13.) We strongly recommend going to “Circle-line construction” in www.fenomec.unam.mx/ipolygons and playing with different side-lengths.

If $(s-a)(s-b)(s-c)(s-d)-abcd>0$ then $\mathcal{A}_{min}$ is given by equation (7) otherwise, $\mathcal{A}_{min}=0$. The coordinates of the vertices of the quadrilateral with minimum area are obtained by letting $\mathcal{A}_c=\mathcal{A}_{min}$ in the circle-line construction.

As far as we know there is no known formula for the minimum area of a polygon with more than four sides and self-intersections.

[1]If $a=c$ and $b=d$ then $P=0$ and there is an infinite number of configurations with area 0.

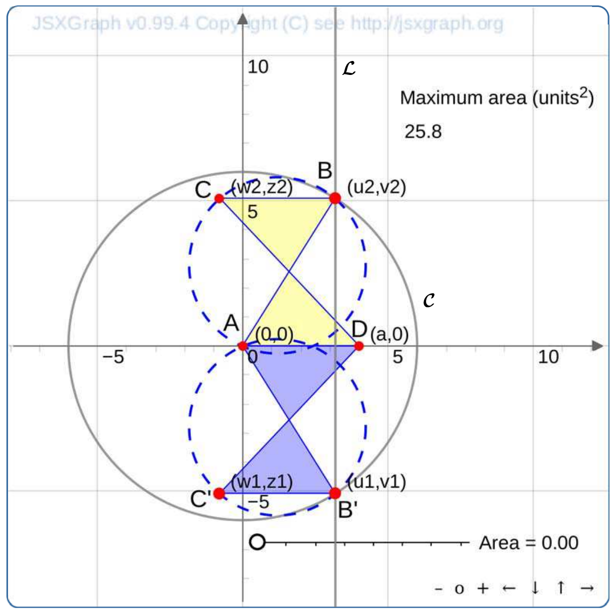

Figure 12. Two configurations with area zero of bow-tie with side-lengths 4,6,4,7. These minima are not cyclic.

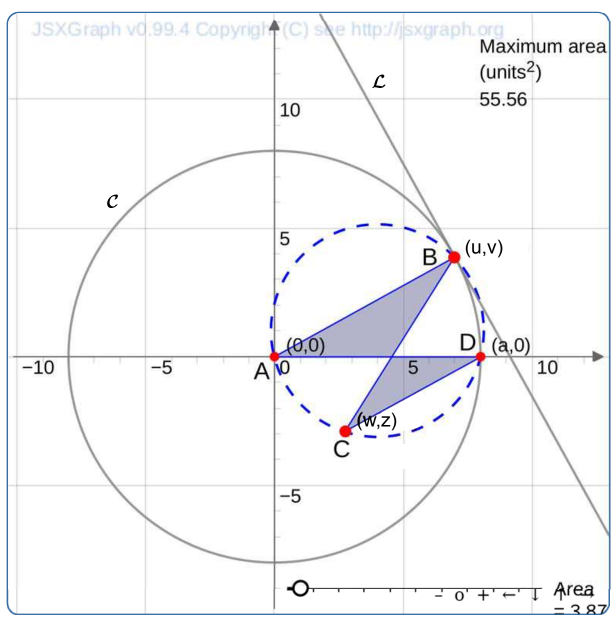

Figure 13. Bow-tie with side-lengths 8,8,8,6 and minimum area. This configuration is cyclic.

4. Polygonal shapes determined by angles and areas

In this section we consider a polygon with fixed side-lengths and study how its shape depends on the prescribed area.

4.1. *Case n=4.* Again the method from [6] described in Section 2 provides the coordinates of the vertices of the quadrilateral with any prescribed area between the maximum and the minimum. You can view an animation of this under “Circle-line construction (movie)” in www.fenomec.unam.mx/ipolygons.

Another way to compute the possible areas and shapes of a quadrilateral if coordinates are not needed is by using angle-area graphs. Referring to Figure 1, if angle α is given then by the cosine rule:

$$a^2 + b^2 - 2ab\cos\alpha = \overline{BD}^2 = c^2 + d^2 - 2cd\cos\gamma$$

and therefore

$$\gamma(\alpha) = \arccos\left(\frac{-a^2 - b^2 + c^2 + d^2 + 2ab\cos\alpha}{2cd}\right), \tag{8}$$

where the inverse cosine can be in the principal or secondary branch.

Renaming constants and using (8), (1) can be written as

$$\mathcal{A}(\alpha) = \sqrt{C_1 - C_2\cos^2\left(\frac{\alpha+\gamma(\alpha)}{2}\right)}. \tag{9}$$

where $C_1 = (s-a)(s-b)(s-c)(s-d)$ and $C_2 = abcd$.

To find the shape if side-lengths and area are given we can plot α *vs* area from equation (9) and read the values of the angle for a given area. We show examples of this in Figures 14 and 15, where we use Bretschneider, unoriented area.

The first hump of Figure 14 corresponds to α between 0 and π and the second to α between π and 2π. Points I, III, IV correspond to quadrilaterals in the upper half-plane; points II, V, VI to quadrilaterals in the lower half-plane. Points VII and VIII correspond to degenerate quadrilaterals. There are two possible configurations for any area between the minimum and the maximum in each half-plane or equivalently, two with clockwise orientation and two with counterclockwise orientation.

Figure 15 shows only values of α between 0 and π. Again each point on the angle-area graph corresponds to a different shape. Observe again that for a fixed orientation there is a unique shape with maximum and minimum area and two different shapes for areas in between. Also, there are two possible areas for each value of angle α; this is because angle γ can have two possible values depending on the branch of the inverse cosine.

If a more accurate value of α is required one can solve for α from equation (9) given a prescribed area $\mathcal{A}$ to obtain

$$\alpha + \gamma(\alpha) = 2\arccos\left(\sqrt{\frac{C_1 - \mathcal{A}^2}{C_2}}\right).$$

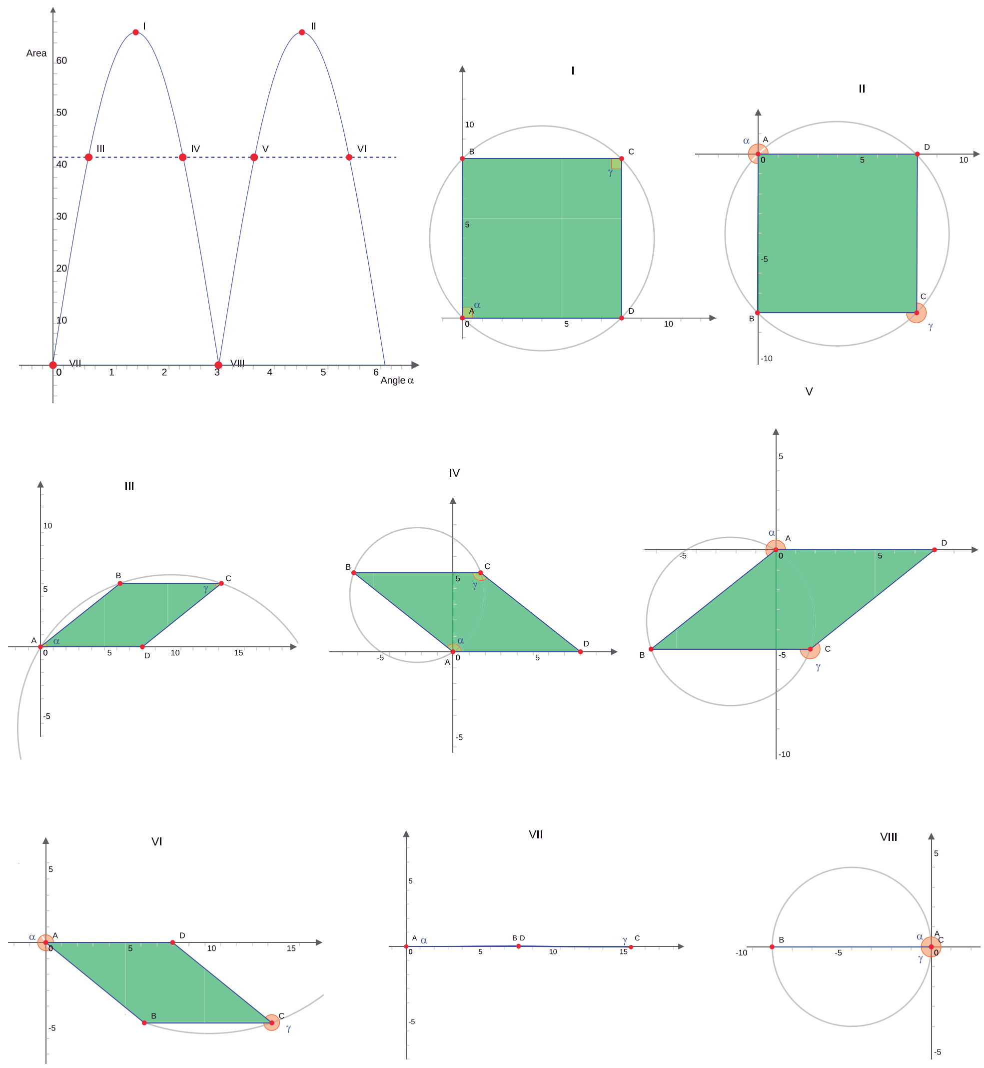

Figure 14. Angle-area graph and corresponding shapes for quadrilateral with side-lengths 8,8,8,8. Figures I and II correspond to the shapes with maximum area, I with sides in clockwise order and II with sides in counterclockwise order. Figures III to VI have the same area, III and VI in clockwise order, V and VI in counterclockwise order. Figures VII and VIII are degenerate cases with area 0.

and then use a Newton method to solve for α.

If we use oriented (Shoelace) area, angle-area graphs have a positive and a negative region. The negative region comes from changing the orientation of the sides. In some quadrilaterals it is possible to go from one orientation to the other by continuously deforming the shape, without reflecting. In others it is not.

This is best studied by identifying 0 and 2π and thus working on the cylinder. If the angle-area graph is disconnected on the cylinder then the two orientations are

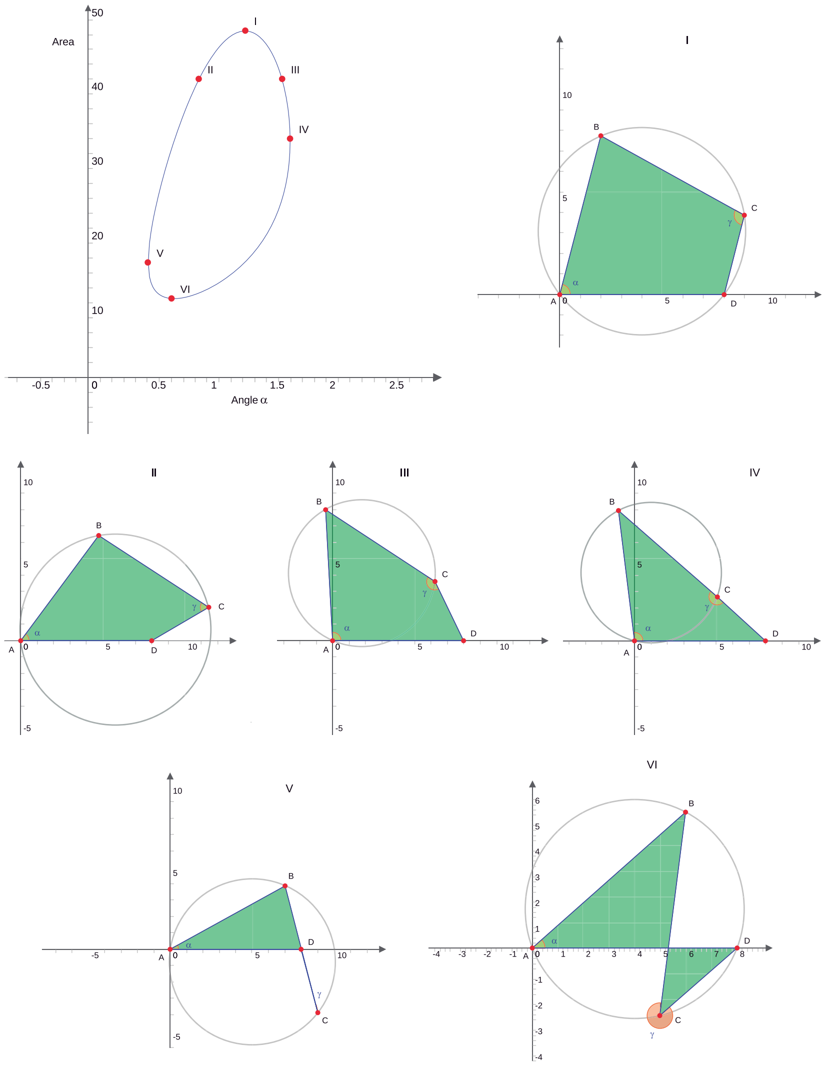

Figure 15. Angle-area graph and shapes for quadrilateral with side-lengths 8,8,8,4. Figure I show the configuration with maximum area; figure V a configuration with minimum area without self-intersection and figure VI the configuration with minimum area and self-intersection.

disjoint. Angle-area graphs on the cylinder provide a way of classifying quadrilaterals into four different classes depending on whether the graph consists of (i) one curve without self-intersection, (ii) one curve with self-intersection, (iii) two curves not homotopic to 0, (iv) two curves homotopic to 0. We show examples in Figure 16. In the left-most figure we used the same side-lengths as in Figure 14 but now the second hump has negative area. In this case it is possible to go from one orientation to the other by deforming continuously.

In the right-most figure we used the same side-lengths as in Figure 15. It is not possible to go from one orientation to the other by deforming continuously.

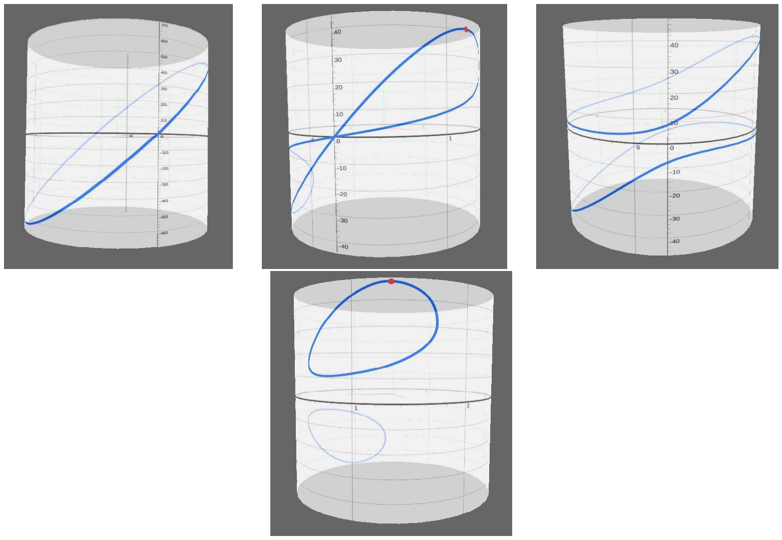

Figure 16. Angle-area graphs on the cylinder showing the four possible classes. From left to right side-lengths 8,8,8,8; 8,7,6,5; 8,5,8,6 and 8,8,8,4.

4.2. *Case n=5.* One can decompose a pentagon into a quadrilateral and a triangle and therefore there are two free angles plus the branches of the inverse cosine. The angle-area graph is now a surface and $\mathcal{A}_c = constant$ is a plane and therefore the curve of intersection between the two corresponds to an infinite number of configurations with the same side-lengths and the same area. In Figure 17 we show a portion of an angle-area graph located near the maximum area. It would be interesting to study complete angle-area graphs of pentagons.

4.3. *Other polygons.* For $n > 4$ there is an infinite number of configurations with any area between the minimum and the maximum. There is no formula for the shape but we wrote a program using JSXGraph that allows one to change the shape and compute the resulting area. We start with the configuration with maximum area, as computed numerically in Section 1. The user can then deform the shape and the area is computed from the Shoelace formula (go to “Build your own polygon” in www.fenomec.unam.mx/ipolygons). In Figure 18 we show an example of a polygon with ten sides in the configuration with maximum area and three other configurations with the same smaller area.

5. Conclusions

In this paper we study the possible areas and shapes of irregular polygons with prescribed side-lengths. We give a way to compute the maximum possible area of any feasible irregular polygon and draw the shape with that area. For quadrilaterals

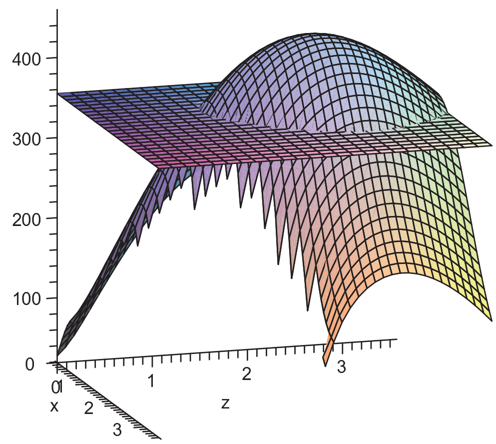

Figure 17. Angle-area graph for a pentagon with side-lengths 10,30,19,12,15 and plane area=355.

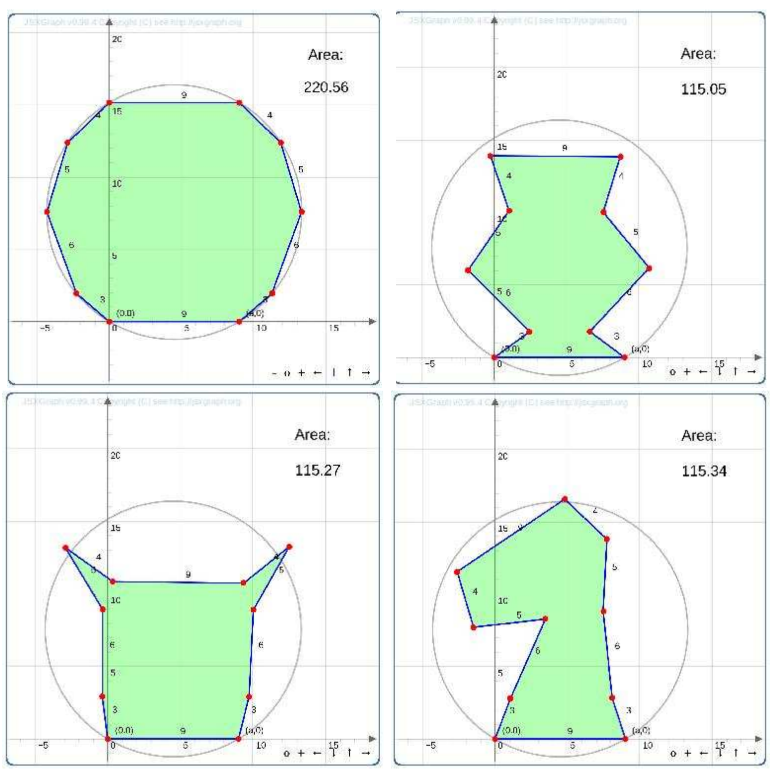

Figure 18. Shape with maximum area for side-lengths 9,3,6,5,4,9,4,5,6,3 units and some other possible shapes with area 115 units squared.

we use the Circle-Line Construction. For general polygons we find the circumcircle of the configuration with maximum area.

To study areas of quadrilaterals with self-intersections we use oriented area. We show that not all minima of quadrilaterals are cyclic and explain the fact both in terms of Bretschneider's Formula and the Circle-Line Construction.

The use of angle-area graphs on the cylinder led to a a classification of quadrilaterals into four different classes which are related to the possibility of reversing orientation by continuously deforming the quadrilaterals.

We provide a computer program in javascript which makes use of JSXGraph to draw the configuration of maximum area of a polygon with given sides. The figure can then be modified to attain any area between the maximum and the minimum.

These results have been useful to us in the study of the codices.

Appendix A: Circle-line construction

Consider Figure 19:

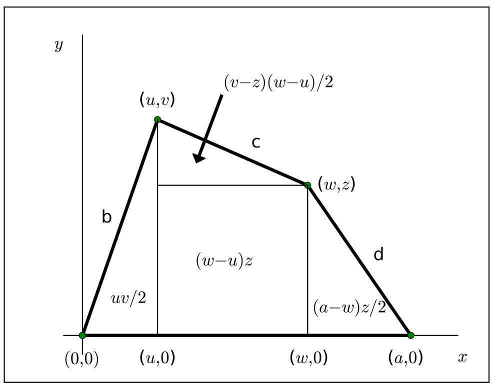

Figure 19. Coordinates of the quadrilateral and area in terms of (u, v) and (w, z) for Appendix A.

By Pythagoras' Theorem, the coordinates of the vertices of the quadrilateral satisfy

$$u^2 + v^2 = b^2, \quad (w - u)^2 + (v - z)^2 = c^2, \quad (a - w)^2 + z^2 = d^2. \qquad (10)$$

The area, $\mathcal{A}_c$, of the quadrilateral using coordinates is calculated subdividing it in triangles and rectangles and is given by

$$2\mathcal{A}_c = vw + z(a - u). \qquad (11)$$

Simplification of equations (10) and (11) give the unknown coordinates (w, z) in terms of (u, v) as

$$w = \frac{M(a-u) + 2\mathcal{A}_c v}{(a-u)^2 + v^2} \quad \text{and} \quad z = \frac{2\mathcal{A}_c(a-u) - Mv}{(a-u)^2 + v^2}, \qquad (12)$$

where $2M = a^2 + c^2 - b^2 - d^2$. Substitution of (12) into (11) gives the coordinates (u, v) as the intersection of a line and a circle given respectively by

$$\mathcal{L} : Pu + Qv = R, \quad \mathcal{C} : u^2 + v^2 = b^2. \tag{13}$$

where

$$\begin{aligned} P &= 2a(a^2 - M - d^2) = a(a^2 + b^2 - c^2 - d^2),\ Q = 4\mathcal{A}_c a, \\ R &= (M - a^2)^2 + a^2 b^2 + 4\mathcal{A}_c^2 - d^2(a^2 + b^2). \end{aligned}$$

Therefore, for fixed side-lengths, the intersection of the line and the circle depends on $\mathcal{A}_c$. If it is too big for the given side-lengths there is no intersection; if it is the maximum possible area, $\mathcal{A}_{max}$ the line is tangent to the circle and if $\mathcal{A}_c$ is smaller then $\mathcal{L}$ intersects $\mathcal{C}$ twice giving two possible coordinates (u_1, v_1) and (u_2, v_2) and thus two different shapes for the given quadrilateral data.

The configuration with maximum area occurs when the line is tangent to the circle. The y coordinate of the intersection of the line and the circle in (13) is

$$v = \frac{R - Pu}{Q} = \sqrt{b^2 - u^2}.$$

This leads to the equation

$$u^2(P^2 + Q^2) + u(-2RP) + R^2 - Q^2 b^2 = 0.$$

The double root of this quadratic gives the $x-$coordinate of vertex B of the quadrilateral and the $y-$ coordinate is found from the equation of the line and therefore the coordinates (u, v) of vertex B are

$$u = \frac{RP}{Q^2 + P^2}, \qquad v = \frac{R}{Q} - \frac{RP^2}{Q(Q^2 + P^2)}, \tag{14}$$

where $\mathcal{A}_c = \mathcal{A}_{max}$. The coordinates (w, z) of vertex C are

$$w = \frac{M(a - u) + 2\mathcal{A}_{max} v}{a^2 + b^2 - 2au}, \ \ z = \frac{2A_{max}(a - u) - Mv}{a^2 + b^2 - 2au}. \tag{15}$$

Appendix B: Proof that either F_1 or F_2 have a root

Recall that

$$F_1(r) = \sum_{j=1}^{n} \text{Arccos}(1 - \frac{a_j^2}{2r^2}) - 2\pi \tag{16}$$

and

$$F_2(r) \;=\; \sum_{j \neq k}^{n} \text{Arccos}(1 - \frac{a_j^2}{2r^2}) - \text{Arccos}(1 - \frac{a_k^2}{2r^2}) \tag{17}$$

where a_k is the longest side and that r_l is the smallest positive r for which F_1 and F_2 are defined.

The derivative of F_1 is given by $\sum_j \frac{-2a_j}{r\sqrt{4r^2 - a_j^2}}$ and is therefore negative for all r. Since $\lim_{r\to\infty} F_1(r) = -2\pi$ then F_1 will have a root as long as $F_1(r_l) > 0$. In this case the center of the circumcircle is inside the polygon of maximum area.

If $F_1(r_l) < 0$ then we will show first that $F_2(r_l) < 0$ and then that F_2 has a root if the side-lengths satisfy the compatibility condition. Since $F_1(r_l) < 0$ then

$$\sum_{j=1}^{n} \text{Arccos}(1 - \frac{a_j^2}{2r_l^2}) < 2\pi$$

but

$$\sum_{j=1}^{n} \text{Arccos}(1 - \frac{a_j^2}{2r_l^2}) \quad = \quad F_2(r_l) + 2\text{Arccos}(1 - \frac{a_k^2}{2r_l^2})$$

therefore $F_2(r_l) < 2\pi - 2\text{Arccos}(1 - \frac{a_k^2}{2r_l^2}) = 0$, since $r_l = a_k/2$ and $\text{Arccos}(-1) = \pi$.

Using the asymptotic expansion in [1],

$$\arcsin(1 - z) = \frac{\pi}{2} - (2z)^{1/2}[1 + \sum_{m=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2^{2m}(2m+1)m!} z^m],$$

since $\arccos(x) = \pi/2 - \arcsin(x)$, taking $z = a_j^2/2r^2$ and keeping only the first term in z we obtain

$$\lim_{r\to\infty} F_2 \simeq \sum_{j\neq k}^{n} \frac{a_j}{r} - \frac{a_k}{r}$$

which is positive if $a_k < \sum_{j\neq k} a_j$, therefore F_2 will have at least one root when the side-lengths satisfy the compatibility condition.

Appendix C: Proof that quadrilaterals with self-intersection are cyclic if an only if opposite angles add up to 2π radians

Referring to Figure 20 (left), if the quadrilateral is cyclic then since angles α and $2\pi - \gamma$ are subtended by the same arc BD they are equal and therefore $\alpha + \gamma = 2\pi$.

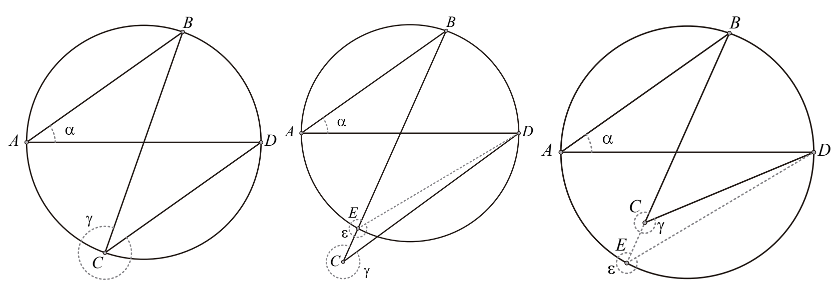

Figure 20. Cyclic bow-tie.

On the other hand, if opposite angles add up to 2π radians then we show by contradiction that the quadrilateral must be cyclic. Draw the circumcircle of ABD, see Figure 20 (middle and right). If vertex C is not on the circle then call E the intersection of the line BC with the circle. By hypothesis $\alpha + \gamma = 2\pi$ but $\alpha + \varepsilon$ also equals 2π because $ABED$ is cyclic. Therefore C has to coincide with E.

References

[1] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions*, Dover, New York (1970).

[2] Böröczky, K., The minimum area of a simple polygon with given side lengths, *Periodica Mathematica Hungarica*, 39, Numbers 1-3 (2000) 33–49.

[3] Codex Vergara, Gallica, gallica.bnf.fr searching for Codex Vergara or directly at gallica.bnf.fr/ark:/12148/btv1b84528032/f21.item.r=codex vergara

[4] H. Demir, Maximum area of a Region Bounded by a Closed Polygon with Given Sides, *Math. Mag.*, 39 (1966) 228–231.

[5] C. E. Garza-Hume, M. C. Jorge y Jorge, and A. Olvera, Quadrilaterals and Bretschneider's Formula, submitted to *Mathematics Teacher*.

[6] M. C. Jorge y Jorge, B. Williams, C. E. Garza-Hume, and A. Olvera, Mathematical accuracy of Aztec land surveys assessed from records in the Codex Vergara, *PNAS*, 108 (2011) no. 37.

[7] B. Polster, *The Shoelace Book*, AMS, 2006.

[8] D. P. Robbins Areas of Polygons Inscribed in a Circle, *Discrete Comput. Geom.*, 12 (1994) 223–236.

[9] B. J. Williams, and H. Harvey, *The Códice de Santa María Asunción: Households and Lands in Sixteenth Century Tepetlaoztoc*, Salt Lake City, USA. University of Utah Press (1997).

[10] B. J. Williams, and F. Hicks, *El Códice Vergara. Edición Facsimilar con Comentario: pintura indígena de casas, campos y organización social de Tepetlaoztoc a mediados del siglo XVI.* Mexico City, Mexico. Universidad Nacional Autónoma de México & ADABI de México A.C. (2011)

C. E. Garza-Hume: IIMAS, UNAM, Circuito Escolar, CU, CDMX 04510, MEXICO
E-mail address: `clara@mym.iimas.unam.mx`

Maricarmen C. Jorge: IIMAS, UNAM, Circuito Escolar, CU, CDMX 04510, MEXICO
E-mail address: `mcj@mym.iimas.unam.mx`

Arturo Olvera: IIMAS, UNAM, Circuito Escolar, CU, CDMX 04510, MEXICO
E-mail address: `aoc@mym.iimas.unam.mx`