Forum Geometricorum
Volume 18 (2018) 39–42.

FORUM GEOM
ISSN 1534-1178

Two Hinged Regular n-sided Polygons

Manfred Pietsch

Abstract. We consider two regular polygons of the same type hinged at a common vertex C. If we further generate two regular polygons of the same type, each on a segment with one neighboring vertex of C from each polygon as endpoints, then these two new polygons must have a common center. The proof is based on two propositions about properties of equidiagonal quadrilaterals.

Given any two regular n-sided polygons hinged together at a common vertex C. Let A and B (respectively D and E) be the adjacent vertices of C in both polygons.

Theorem 1. *The regular n-sided polygons over the line segments BD and AE have a common center* (see Figure 1).

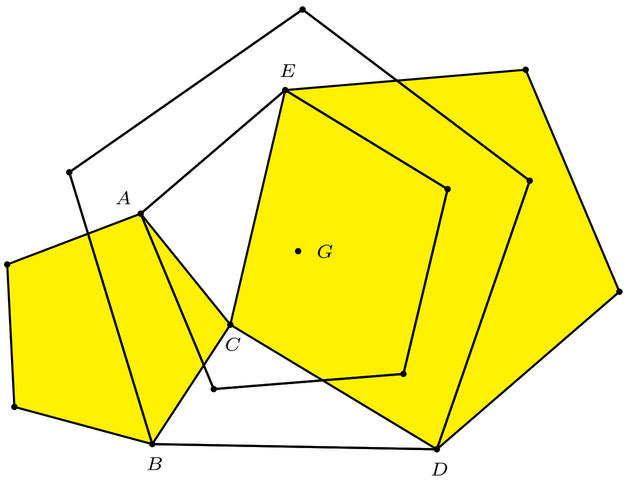

Figure 1.

Proposition 2. *Given two similar isosceles triangles ABC and DEC hinged at the vertex C,*

(1) $\overline{BE} = \overline{AD}$,

(2) *the circumferences of the triangles ABC and DEC concur at the intersection point of AD and BE* (see Figure 2).

Publication Date: January 24, 2018. Communicating Editor: Paul Yiu.

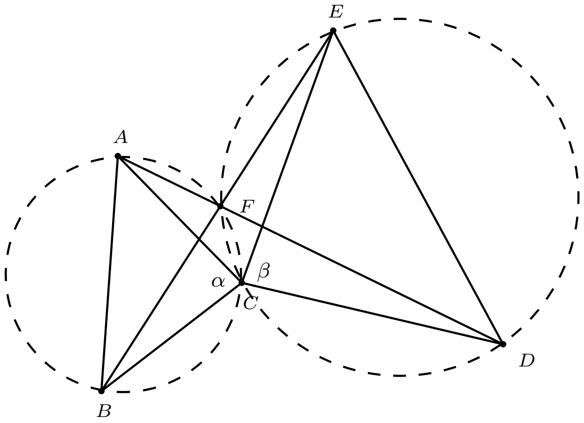

Figure 2.

Proof. If BE and AD intersect at C, the statements (1) and (2) are evident.

Let therefore BE and AD intersect at a point $F \neq C$.

(1) The triangles ACD and BCE are congruent:

- $AC = BC$ and $CD = CE$,
- $\angle DCA = \angle ECB$ as $\alpha = \beta$.

Thus, $BE = AD$.

(2) The congruence implies that the triangle BCE can be interpreted as the result of a rotation of triangle ACD around C through α. Therefore the angles between BE and AD must be identical to α: $\angle AFB = \angle DFE = \alpha (= \beta)$.

$\angle AFB = \alpha$ implies that C and F are located on the circumcircle of triangle ABC. Analogously, C and F are also on the circumcircle of triangle DEC. Thus F is located on both circumferences, which was to be proved. □

Proposition 3. *Given a quadrilateral $ABDE$ whose diagonals AD and BE are of equal length and intersect at a point F. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the circumcircles of triangles AFE and BDF respectively. The perpendicular bisectors of BD and AE intersect $\mathcal{K}_1$ and $\mathcal{K}_2$ at the same point* (see Figure 3).

Proof. If $\mathcal{K}_1$ and $\mathcal{K}_2$ touch each other at point F, then the above statement is evident.

Let therefore $\mathcal{K}_1$ and $\mathcal{K}_2$ intersect at F and another point G (see Figure 4).

The triangles GAD and GEB are congruent:

- $AD = BE$ (given),
- $\angle GBE = \angle GDA$ as both angles are peripheral angles of the chord FG on $\mathcal{K}_2$,
- analogously, $\angle BEG = \angle DAG$ (peripheral angles over FG on $\mathcal{K}_1$).

The congruence implies that $BG = DG$ and $EG = AG$, which was to be proved. □

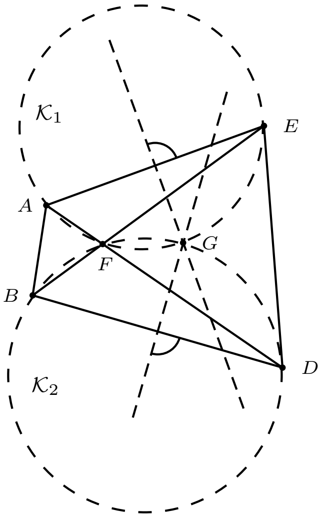

Figure 3

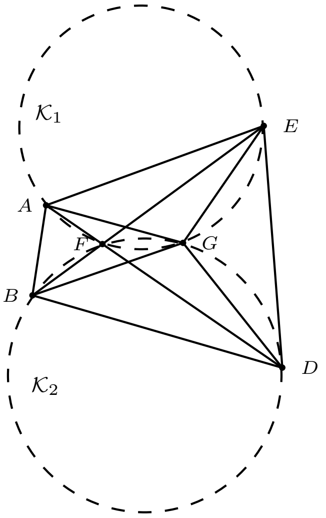

Figure 4

Proof of Theorem 1. In any two regular n-sided polygons hinged together at a common vertex C, the adjacent vertices of C in both polygons form with C two similar isosceles triangles ABC and DEC with $\angle ACB = \angle DCE$ (respectively $\alpha = \beta$); see Figure 5.

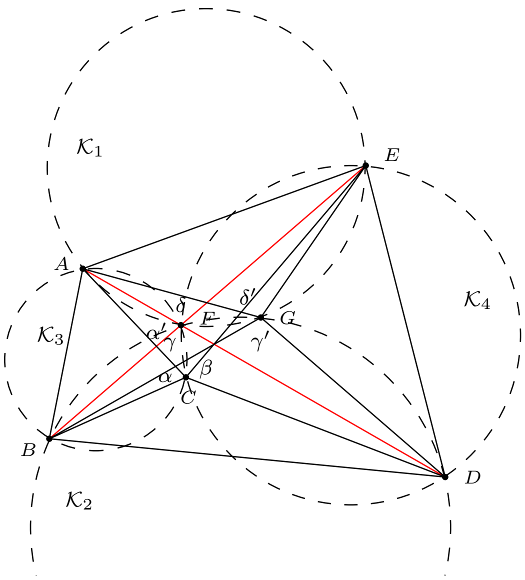

Figure 5.

According to Proposition 2 the quadrilateral $ABDE$ has two diagonals AD and BE with equal length and intersecting at a point F lying on both circumcircles of ABC and EDC. Thus the inscribed angles α and $\alpha'(=\angle AFB)$ must be equal. This implies $\gamma = 180° - \alpha = \delta$, wherein $\gamma = \angle BFD$ and $\delta = \angle EFA$.

As γ and γ' (respectively δ and δ') are inscribed angles in $\mathcal{K}_2$ (respectively $\mathcal{K}_1$), we obtain

$$\gamma' = 180° - \alpha = \delta' \tag{$*$}$$

Proposition 3 guarantees that the triangles BDG and EAG are isosceles. Therefore, they are similar isosceles triangles.

Because of (*) the interior angles α (respectively β) of the original polygons fit to the central angles γ' (respectively δ') of the generated polygons. These must also be regular n-sided polygons and have the common center G. □

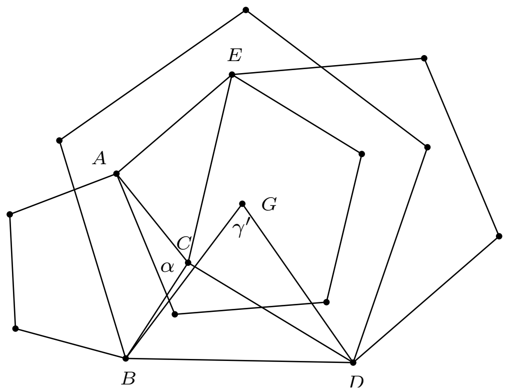

Figure 6.

Manfred Pietsch: Stiftisches Gymnasium, Altenteich 14, 52349 Düren, Germany
E-mail address: manfredpietsch@googlemail.com