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Revisiting the Infinite Surface Area of Gabriel's Horn

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Abstract. We show that the integral which gives the surface area of Gabriel's horn can be calculated in a simple way, thus eliminating the need for a comparison theorem to prove its divergence.

Gabriel's horn is defined as the solid obtained by revolving the region

$$\mathcal{R} = \left\{ (x,y) : \ x \in [1, \ +\infty), 0 \le y \le \frac{1}{x} \right\}$$

about the x-axis. This object has been of enduring interest because is has the curious property of having finite volume, yet infinite surface area. A straightforward application of the disk method easily shows the horn's volume to be equal to π. (An interesting "wedding cake" version has been considered by Julian Fleron [2], the volume of which is $\frac{1}{6}\pi^3$). Regarding the surface area S of Gabriel's horn, one easily sees that it is given by the integral

$$S = 2\pi \int_1^\infty \frac{\sqrt{1+x^4}}{x^3} dx, \tag{1}$$

which needs to be proven divergent. The standard approach found in many books (see e.g. [1]) is to use comparison properties of improper integrals, and to show that

$$\int_1^\infty \frac{\sqrt{1+x^4}}{x^3} dx \ge \int_1^\infty \frac{1}{x} dx. \tag{2}$$

In his paper, Fleron mentions that the integral in (1) "cannot be evaluated readily", notes that it can be solved with a computer algebra system, and proceeds to perform the usual estimate (2). The purpose of our short note is to show that (1) can in fact be solved quite easily, thereby providing a direct evaluation of S:

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Take $2\int \frac{\sqrt{1+x^4}}{x^3}dx$ and make the substitution $x^2 = \tan\theta$. Then $2xdx = \sec^2\theta d\theta$, and the integral becomes

$$\begin{aligned}\int \frac{\sec\theta}{\tan^2\theta}\sec^2\theta d\theta &= \int \sec\theta\csc^2\theta d\theta \qquad \text{integration by parts}\\ &= -\sec\theta\cot\theta + \int \sec\theta\tan\theta\cot\theta d\theta\\ &= -\csc\theta + \ln|\sec\theta+\tan\theta| + C.\end{aligned}$$

It is now obvious that

$$2\pi\int_1^\infty \frac{\sqrt{1+x^4}}{x^3}dx = \pi\left[-\csc\theta + \ln|\sec\theta+\tan\theta|\right]_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} = \infty. \qquad (3)$$

Returning to the original variable gives the solution:

$$2\int \frac{\sqrt{1+x^4}}{x^3}dx = \ln\left[\sqrt{x^4+1}+x^2\right] - \frac{\sqrt{x^4+1}}{x^2} + C, \qquad (4)$$

and of course we see again that $2\pi\int_1^\infty \frac{\sqrt{1+x^4}}{x^3}dx = \infty$. In addition, the solved integral allows for the calculation of the surface area of a finite piece of Gabriel's horn, defined on any $[a,\ b] \subset [1,\ \infty)$.

References

[1] E. Ellis and D. Gulick, *Calculus: One and Several Variables*, Harcourt Brace Jovanovich, Fort Worth, 1991.

[2] J. Fleron, Gabriel's wedding cake, *College Math. J.*, 30 (1999) 35–38.

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