Forum Geometricorum
Volume 18 (2018) 47–55.

FORUM GEOM
ISSN 1534-1178

Integer Sequences and Circle Chains Inside a Circular Segment

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Abstract. We derive the conditions for inscribing, inside a circular segment, a chain of mutually tangent circles having the property that the ratio between the largest circle radius and the radius of any other circle of the chain is an integer number.

1. Introduction

If we consider a straight line intersecting a circle $\mathcal{C}$, the two areas bounded by the common chord and one of the two arcs are named circular segments. Inside each one of the circular segments it is possible to inscribe infinite chains of mutually tangent circles which are also tangent to the outer circle and to the common chord; a generic example of chain is shown in Figure 1.

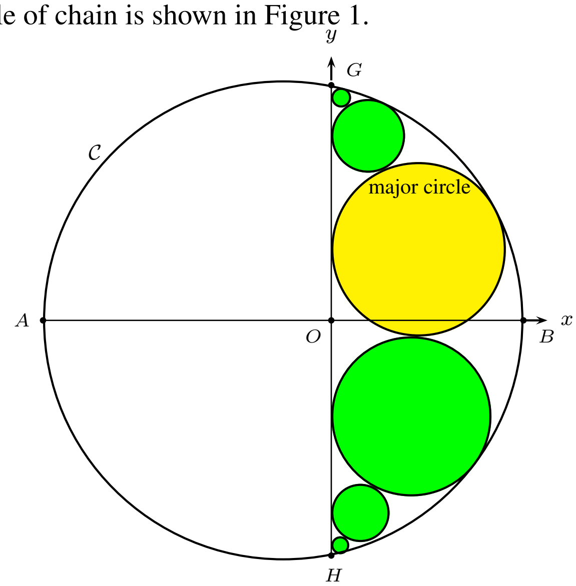

Figure 1. Example of a circle chain inscribed inside a circular segment

In a previous paper [2], we derived some geometrical properties and formulas related the circles forming the chain. Here, we follow a different approach, based on the inversion technique, and investigate about some connections that can be found between the circle chains and certain integer sequences.

Publication Date: January 26, 2018. Communicating Editor: Paul Yiu.

2. Some definitions and useful expressions

For the following, it is convenient to define the *major circle* in the chain (see Figure 1) as the one having the largest radius and label it by index 0; thus, we can subdivide a generic chain into two sub-chains: an up chain starting from the major circle and converging to point G and a down chain starting from the major circle and converging to point H.

Moreover, let $2(a+b)$ be the diameter of the outer circle and $2b$ the length of segment OB. By setting up a coordinate system with origin at O, we have A, B, G and H with coordinates $(-2a, 0)$, $(2b, 0)$, $(0, 2\sqrt{ab})$ and $(0, -2\sqrt{ab})$ respectively.

In this paper, we want to investigate the conditions under which the ratios τ_k, $k = 1, 2, \dots$, between the radius of the major circle and the generic k-th circle are integers for both the up and down sub-chains. In other words, what are the conditions (provided they exist) for the radii of the circles of the chain to be submultiples of the major circle radius?

From [2], we report that the radius r_i and center coordinates $(X_i,\ Y_i)$ are related by the following formula:

$$r_i = X_i = -\frac{Y_i^2}{4a} + b, \qquad i = 0,\ \pm 1,\ \pm 2,\ \dots. \tag{1}$$

Note that, for the major circle ordinate Y_0, the following relation must hold:

$$-2a\left(-1 + \sqrt{1 + \frac{b}{a}}\right) \le Y_0 \le 2a\left(-1 + \sqrt{1 + \frac{b}{a}}\right). \tag{2}$$

In correspondence of two particular values for Y_0, we have two symmetrical dispositions for the up and down chains:

- if $Y_0 = 0$, we have that $r_0 = b$ and the major circle is bisected by the x-axis (central symmetry);
- if $Y_0 = \pm 2a\left(-1 + \sqrt{1 + \frac{b}{a}}\right)$, we have that $r_0 = 2a\left(-1 + \sqrt{1 + \frac{b}{a}}\right)$ and two equal major circles, (one for the up chain and one for the down chain), both tangent to x-axis, exist (bi-central symmetry).

3. Circle radii expressions by inversion

In [2], the expressions for the center coordinates (X_i, Y_i) and radius r_i of the generic i-th circle of the chain have been obtained in form of continued fractions. Here we want to get them in a closed form. According to a hint of F. J. García Capítan (in a personal communication), such a result can be obtained by means of the inversion technique.

In particular, we are interested in obtaining a closed form formula for the ratio between the major circle radius and the generic k-th circle radius.

To this aim, it is worthwhile to remark some points and to write some equations that shall be used in the following:

- Outer circle $\mathcal{C}$ equation:

$$x^2 + y^2 - 2(b-a)x - 4ab = 0 \tag{3}$$

- Inversion circle $\mathcal{C}_{\text{inv}}$ (center in H and radius HG) having equation:
$$x^2 + y^2 + 4\sqrt{ab}y - 12ab = 0 \tag{4}$$
and center coordinates $(X_{\text{inv}},\ Y_{\text{inv}})$ and radius R_{inv} given respectively by:
$$X_{\text{inv}} = 0, \quad Y_{\text{inv}} = -2\sqrt{ab}, \qquad R_{\text{inv}} = 4\sqrt{ab}. \tag{5}$$

- The inversive image of the y-axis with respect to $\mathcal{C}_{\text{inv}}$ is still the y-axis while the image of the outer circle $\mathcal{C}$ is the straight line s:
$$(a-b)x - 2\sqrt{ab}y + 4ab = 0. \tag{5}$$
The straight line s is the radical axis relevant to the outer circle C and to the inversion circle $\mathcal{C}_{\text{inv}}$ (see Figure 2). For convenience, in Figure 2 we have also added the bisector β of the angle between the y-axis and the straight line s.

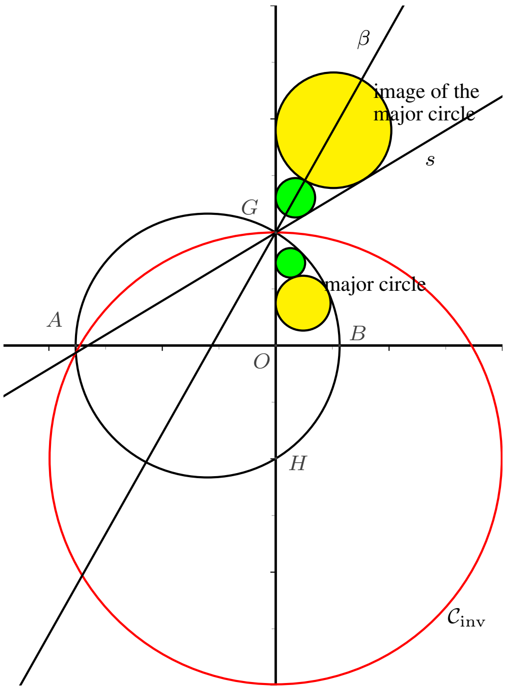

Figure 2. Some circles of the chain with their inversive images

- The cosine of the angle ψ between the y-axis and the straight line s is given by:
$$\cos\psi = \frac{a-b}{a+b} \tag{6}$$
and from (6) one can deduce:
$$\sin\frac{\psi}{2} = \sqrt{\frac{b}{a+b}}. \tag{7}$$

Now, let us consider, the major circle having center coordinates and radius given respectively by:

$$(X_0,\ Y_0) = \left(-\frac{Y_0^2}{4a} + b,\ Y_0\right), \tag{8a}$$

$$r_0 = -\frac{Y_0^2}{4a} + b. \tag{8b}$$

The inversive image of the major circle, by applying the formulas given in [4], is still a circle having center coordinates and radius given by:

$$(X_0',\ Y_0') = \left(\frac{16ab}{(Y_0 + 2\sqrt{ab})^2}\left(-\frac{Y_0^2}{4a} + b\right),\ -2\sqrt{ab} + \frac{16ab}{Y_0 + 2\sqrt{ab}}\right), \tag{9a}$$

$$r_0' = \frac{16ab}{(Y_0 + 2\sqrt{ab})^2}\left(-\frac{Y_0^2}{4a} + b\right). \tag{9b}$$

The inversive images of the circles inscribed in the circular segment are still mutually tangent circles that are also tangent to the y-axis and to the straight line s (see Figure 2); moreover, their centers lie on the bisector β. In Figure 3, we show three among these circles: the image of the major circle $\mathcal{C}_0'$, the image of the first circle of the up chain $\mathcal{C}_1'$ and the image of the first circle of the down chain $\mathcal{C}_{-1}'$.

In particular by looking at Figure 3, we can easily show, by similitude, that the ratio between the radius of the generic circles $\mathcal{C}_i'$ and the radius of the image of the major circle $\mathcal{C}_0'$ is:

$$\frac{r_k'}{r_0'} = \left(\frac{1 - \sin\frac{\psi}{2}}{1 + \sin\frac{\psi}{2}}\right)^i. \tag{10}$$

From formula (10) and by taking into account (7) and (9b), one has:

$$r_i' = X_i' = \left(\frac{1 - \sqrt{\frac{b}{a+b}}}{1 + \sqrt{\frac{b}{a+b}}}\right)^i \frac{16ab}{(Y_0 + 2\sqrt{ab})^2}\left(-\frac{Y_0^2}{4a} + b\right), \tag{11a}$$

$$Y_i' = -\frac{a}{\sqrt{ab}}\left(\frac{1 - \sqrt{\frac{b}{a+b}}}{1 + \sqrt{\frac{b}{a+b}}}\right)^i \frac{(Y_0 - 2\sqrt{ab})4b}{Y_0 + 2\sqrt{ab}} + 2\sqrt{ab}. \tag{11b}$$

From [4], one can write the following relation:

$$\frac{r_0}{r_i} = \frac{r_0}{r_i'}\left|\frac{(X_i' - X_{\mathrm{inv}})^2 + (Y_i' - Y_{\mathrm{inv}})^2 - (r_i')^2}{R_{\mathrm{inv}}}\right|. \tag{12}$$

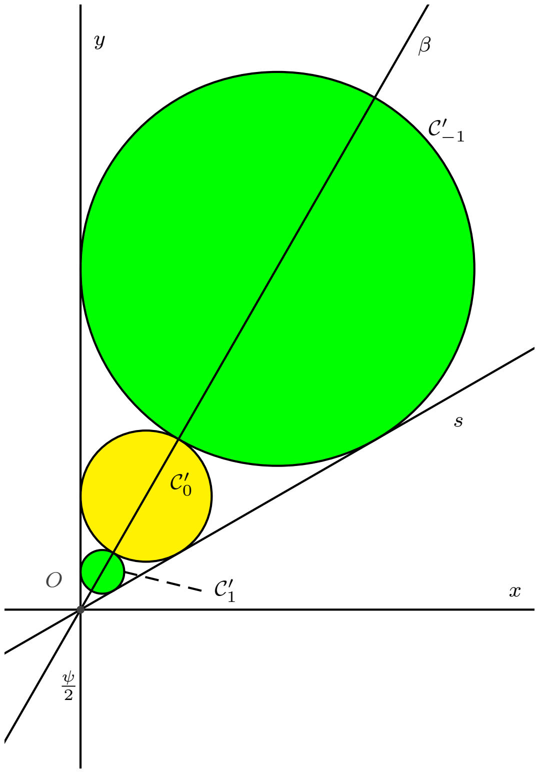

Figure 3. Inversive images of circles $\mathcal{C}_{-1}$, $\mathcal{C}_0$ and $\mathcal{C}_1$

Such a formula, by taking into account (4b), (11a) and (11b), and after some algebra, becomes:

$$\frac{r_0}{r_i} = \left(\frac{1-\sqrt{\frac{b}{a+b}}}{1+\sqrt{\frac{b}{a+b}}}\right)^{i} \frac{(Y_0-2\sqrt{ab})^2}{16ab} + \left(\frac{1-\sqrt{\frac{b}{a+b}}}{1+\sqrt{\frac{b}{a+b}}}\right)^{-i} \frac{(Y_0+2\sqrt{ab})^2}{16ab} - \frac{2(Y_0^2-4ab)}{16ab} \tag{13a}$$

or equivalently,

$$\frac{r_0}{r_i} = \left(\left(\frac{1-\sqrt{\frac{b}{a+b}}}{1+\sqrt{\frac{b}{a+b}}}\right)^{i/2} \frac{(Y_0-2\sqrt{ab})}{4\sqrt{ab}} - \left(\frac{1-\sqrt{\frac{b}{a+b}}}{1+\sqrt{\frac{b}{a+b}}}\right)^{-i/2} \frac{(Y_0+2\sqrt{ab})}{4\sqrt{ab}}\right)^2. \tag{13b}$$

Equation (13b) simply shows that the ratio r_0/r_i is always a square.
Equations (13a) or (13b) define the sequence $\{\tau_i\} = \{r_0/r_i\}$ for $i = 0, \pm 1, \pm 2, \ldots$
On this sequence we shall focus our attention in the following sections.

4. A recursive formula for $\{\tau_k\}$

If we look at (13a), we note that it is a Binet-like formula of the type:

$$\tau_i = A\omega^i + B\omega^{-i} + C \tag{14}$$

where, in our case, is:

$$\omega = \frac{1 - \sqrt{\frac{b}{a+b}}}{1 + \sqrt{\frac{b}{a+b}}}, \tag{15}$$

$$A = \frac{(Y_0 - 2\sqrt{ab})^2}{16ab}, \qquad B = \frac{(Y_0 + 2\sqrt{ab})^2}{16ab}, \qquad C = -\frac{2(Y_0^2 - 4ab)}{16ab}. \tag{16}$$

In [1], J. Kocik has demonstrated that a generic Binet-like relation can be expressed in an equivalent way by means of a non-homogeneous recursive relation of the type:

$$\tau_{i+2} = \left(\omega + \frac{1}{\omega}\right)\tau_{i+1} - \tau_i + C\left(2 - \omega - \frac{1}{\omega}\right) \tag{17}$$

that, by taking into account (15) and the expression for C in (16), becomes:

$$\tau_{i+2} = 2\left(1 + 2\frac{b}{a}\right)\tau_{i+1} - \tau_i + \left[\frac{1}{2}\left(\frac{Y_0}{a}\right)^2 - 2\frac{b}{a}\right]. \tag{18}$$

5. Conditions for $\{\tau_i\}$ to be an integer sequence

In the general case, the sequence $\{\tau_i\}$ is composed of real numbers. Here we want to find the conditions for which the sequence $\{\tau_i\}$ is entirely composed of integers. To this aim, it is convenient to introduce the following variables:

$$u = \frac{Y_0}{a}, \quad v = \frac{b}{a} \quad \text{with the condition } u \leq 2|\sqrt{1+v} - 1|. \tag{19}$$

The condition for the variable u comes from formula (2).

By considering the new variables u and v, equation (13a) becomes:

$$\tau_i = \left(\frac{1 - \sqrt{\frac{v}{1+v}}}{1 + \sqrt{\frac{v}{1+v}}}\right)^i \frac{(u - 2\sqrt{v})^2}{16v} + \left(\frac{1 - \sqrt{\frac{v}{1+v}}}{1 + \sqrt{\frac{v}{1+v}}}\right)^{-i} \frac{(u + 2\sqrt{v})^2}{16v} - \frac{2(u^2 - 4v)}{16v}. \tag{20}$$

From the general point of view, it is possible to impose, by means of (20), that the ratios τ_1 and τ_{-1} are equal to two given real numbers λ and μ ($\lambda > 1$ and $\mu > 1$) respectively; that yields the following system:

$$\begin{cases} \left(\frac{1-\sqrt{\frac{v}{1+v}}}{1+\sqrt{\frac{v}{1+v}}}\right) \frac{(u-2\sqrt{v})^2}{16v} + \left(\frac{1+\sqrt{\frac{v}{1+v}}}{1-\sqrt{\frac{v}{1+v}}}\right) \frac{(u+2\sqrt{v})^2}{16v} - \frac{2(u^2-4v)}{16v} = \lambda, \\ \left(\frac{1+\sqrt{\frac{v}{1+v}}}{1-\sqrt{\frac{v}{1+v}}}\right) \frac{(u-2\sqrt{v})^2}{16v} + \left(\frac{1-\sqrt{\frac{v}{1+v}}}{1+\sqrt{\frac{v}{1+v}}}\right) \frac{(u+2\sqrt{v})^2}{16v} - \frac{2(u^2-4v)}{16v} = \mu \end{cases} \tag{21}$$

having the solution

$$\begin{cases} u &= \sqrt{\lambda} - \sqrt{\mu}, \\ v &= \frac{(\sqrt{\lambda}+\sqrt{\mu})^2}{4} - 1. \end{cases} \tag{22}$$

Thus, if $\lambda = m$ and $\mu = n$, (m, n being integers), we have that by choosing a, b and Y_0 satisfying the relations:

$$\frac{Y_0}{a} = \sqrt{m} - \sqrt{n}, \qquad \frac{b}{a} = \frac{(\sqrt{m}+\sqrt{n})^2 - 4}{4}. \tag{23}$$

we also have that:

$$\tau_1 = m, \qquad \tau_{-1} = n. \tag{24}$$

Relations (23) only imply that τ_1 and τ_{-1} are integers, but do not imply that τ_i and τ_{-i}, $i = 2, 3, \dots$ are also integers.

Nevertheless, if m and n are integers satisfying the following relation:

$$mn = K^2 \tag{25}$$

(that is their product is equal to a square integer) we have that the coefficients of the recursion (18) are always integer numbers and their expressions are:

$$\begin{aligned} 2\left(1 + 2\tfrac{b}{a}\right) &= -2 + m + 2K + n, \\ \tfrac{1}{2}\left(\tfrac{Y_0}{a}\right)^2 - 2\tfrac{b}{a} &= -2K + 2. \end{aligned} \tag{26}$$

Therefore, being $\tau_0 = 1$, $\tau_1 = m$, $\tau_{-1} = n$ and taking into account (26), we have, from (18) that τ_2 and τ_{-2} too are integers. Finally, due to the fact that the expressions in (26) represent integer numbers and being (18) a recursive relation, we deduce that τ_i is an integer for any index i, and conclude that (23) together (25) are sufficient conditions in order to be $\{\tau_i\}$ an integer sequence.

6. Symmetrical chains

Two particular cases relevant to conditions (23) and (25) are represented by:

(a) $m = n$.
(b) $m = 1$, $n = K^2$ or $m = K^2$, $n = 1$.

In case (a) one finds the chains with central symmetry while in case (b) the chains with bi-central symmetry.

In both cases the *up* and *down* sequences are equal and some of them are classified in OEIS (The *On Line Encyclopedia of Integer Sequences*) [3]. We reported them in Tables I and II.

Table I: sequences listed in OEIS and related to
chains having central symmetry

Values of m and n	**Classification of the sequence according to** OEIS
$m = n = 2$	$\mathbf{A055997}$
$m = n = 3$	$\mathbf{A171640}$
$m = n = 4$	$\mathbf{A055793}$
$m = n = 9$	$\mathbf{A055792}$
$m = n = 10$	$\mathbf{A247335}$

Table II: sequences listed in OEIS and related to
chains having bi-central symmetry

Values of m and n	**Classification of the sequence according to** OEIS
$m = 1, n = 4$ **or** $n = 1, m = 4$	$\mathbf{A081068}$
$m = 1, n = 25$ **or** $n = 1, m = 25$	$\mathbf{A008844}$
$m = 1, n = 81$ **or** $n = 1, m = 81$	$\mathbf{A046172}$
$m = 1, n = 169$ **or** $n = 1, m = 169$	$\mathbf{A006051}$

7. Examples

Let us show some examples.
(a) Case of non symmetrical chains:
If we choose $m = 2$ and $n = 8$, we have from (23):

$$\frac{Y_0}{a} = -\sqrt{2}, \quad \frac{b}{a} = \frac{7}{2}.$$

The generated integer sequences up and down are respectively:

$$\{\tau_{\text{up}}\} = \ 1,\ 2,\ 25,\ 392,\ 6241,\ \ldots,$$
$$\{\tau_{\text{down}}\} = \ 1,\ 8,\ 121,\ 1922,\ 30625,\ \ldots.$$

(b) Case of chains with central symmetry:
If we choose $m = 2$ and $n = 2$, we have from (23):

$$\frac{Y_0}{a} = 0, \quad \frac{b}{a} = 1.$$

The generated integer sequences up and down are respectively:

$$\{\tau_{\text{up}}\} = \ \{\tau_{\text{down}}\} = 1,\ 2,\ 9,\ 50,\ 289,\ 1682,\ \ldots.$$

This is sequence $\mathbf{A055997}$ in OEIS.
(c) Case of chains with bi-central symmetry:
If we choose $m = 1$ and $n = 4$, we have from (23):

$$\frac{Y_0}{a} = -1, \quad \frac{b}{a} = \frac{5}{4}.$$

The generated integer sequences up and down are respectively:

$$\{\tau_{\rm up}\} = \{\tau_{\rm down}\} = 1,\ 4,\ 25,\ 169,\ 1156,\ \ldots.$$

This is sequence **A081068** in OEIS.

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