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Twins of Hofstadter Elements

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Abstract. In this article the Hofstadter triangles, transversals, points and sectrices are considered and, based on these known elements, new ones closely related to Hofstadter's elements called "*twins*" are defined, analyzed and studied.

1. Introduction

We begin with a scalene acute triangle ΔABC. Let a, b, c be its sides opposite the vertices A, B, C and α, β, γ be its respective angles in radians, where without loss of generality we assume that $\alpha < \gamma < \beta \left(< \frac{\pi}{2}\right)$. Let us consider three pairs of straight semi-lines such that the elements of each pair have origins two of the vertices (poles) and each passes through the other vertex of ΔABC and both elements rotate about their poles in opposite directions (counterclockwise and clockwise, respectively) at constant rates analogous to the angles they sweep. The intersection points of the elements of the three pairs of semi-lines for any $x \in [0, 1]$ define the "*Hofstadter triangles*". It is known that the vertices of the Hofstadter triangles draw straight-lines called *"Hofstadter $x-$transversals"* and the vertices of Hofstadter triangle joined with the opposite vertices of ΔABC are concurrent at a point called "*Hofstadter point H_x*" [1]. One more characteristic of what we call $x-$sectrices (a term borrowed from "*Maclaurin's sectrices*" [2]) as x increases from 0 to 1 is that they pass through some well-known points of the triangle, like the vertices of the Morley triangle [3], the incenter, etc. In this article we prove that the orthogonal projections (simply projections from now on) of the vertices of the Hofstadter points H_x, for a certain $x \in [0, 1]$, onto the opposite sides form a "*twin*" Hofstadter triangle. The vertices of the latter triangle joined with the opposite vertices of ΔABC by straight-line segments, the "*twin*" Hofstadter $x-$transversals, are concurrent at a "*twin*" Hofstadter point O_x. More interesting results are found when the "*twin*" Hofstadter triangles, $x-$transversals and points are analyzed and studied in detail. Moreover, as x increases from 0 to 1 each twin Hofstadter point sweeps an arc of a curve called Hofstadter x-sectrix associated with the point considered.

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2. Twin Hofstadter's triangles, transversals, points

Let the two straight semi-lines of the ordered pair $(\mathcal{E}_{\mathrm{BC1}}, \mathcal{E}_{\mathrm{CB2}})$ have origins the vertices B and C and pass through C and B, respectively, rotate about B counterclockwise and about C clockwise at constant rates proportional to β and γ, so that for any $x \in [0,1]$ the angles swept by them are βx and γx, respectively. (*Note:* The two letters in the subscript denote, for $x = 0$, the side of the triangle on which both elements of the pair lie, the first letter denotes the corresponding origin and pole, and the numbers 1 and 2 denote counterclockwise and clockwise rotation.) Let the two straight semi-lines of $(\mathcal{E}_{\mathrm{BC1}}, \mathcal{E}_{\mathrm{CB2}})$ intersect at the point A_x, a vertex of the Hofstadter triangle for $x \in [0,1]$ whose projection onto BC is A'_x. (*Note:* The points A_0 and A'_0 will be defined later.) Similarly, using a cyclic notation and properties we consider the other two pairs of straight semi-lines $(\mathcal{E}_{\mathrm{CA1}}, \mathcal{E}_{\mathrm{AC2}})$ and $(\mathcal{E}_{\mathrm{AB1}}, \mathcal{E}_{\mathrm{BA2}})$, their corresponding $x-$transversals as B_x and C_x sweep the interior of the triangle ΔABC and also their projections B'_x and C'_x onto the sides CA and AB, respectively.

The various definitions given so far enable us to determine new elements called "*twins*", and prove some properties associated with the particular Hofstadter triangles, the $x-$transversals and the Hofstadter points H_x. First, we present a known proposition for a Hofstadter triangle and then we state and prove a new one for its "*twin*" triangle.

Theorem 1 ([1]). *Under the notation and the assumptions so far let $\Delta A_x B_x C_x$ be the Hofstadter triangle corresponding to a certain $x \in [0,1]$. Then, the straight-line segments AA_x, BB_x, CC_x are concurrent at a point H_x whose trilinear coordinates are*

$$\begin{array}{ll} \frac{\sin(x\alpha)}{\sin((1-x)\alpha)} : \frac{\sin(x\beta)}{\sin((1-x)\beta)} : \frac{\sin(x\gamma)}{\sin((1-x)\gamma)} & \text{for } x \in (0,1), \\ \frac{\alpha}{a} : \frac{\beta}{b} : \frac{\gamma}{c} & \text{for } x = 0, \\ \frac{a}{\alpha} : \frac{b}{\beta} : \frac{c}{\gamma} & \text{for } x = 1. \end{array} \tag{1}$$

Theorem 2. *Under the assumptions of Theorem 1, the vertices of the twin Hofstadter triangle $\Delta A'_x B'_x C'_x$, for a certain $x \in [0,1]$, joined with the corresponding vertices A, B, C of ΔABC define the concurrent straight-line segments (x-transverals) AA'_x, BB'_x, CC'_x. (Note: The point of concurrency is called the twin Hofstadter x-point O_x.)*

Proof. We begin with the proof of the statement for any $x \in (0,1]$. From the right triangles $\Delta BA_xA'_x$ and $\Delta CA_xA'_x$ for which $A_xA'_x$ is a common side we obtain that

$$(A_xA'_x) = (BA'_x)\tan(\beta x), \;\; (A_xA'_x) = (A'_xC)\tan(\gamma x). \tag{2}$$

Solving for (BA'_x) and (A'_xC) we immediately get that

$$a = (BA'_x) + (A'_xC) = \frac{(A_xA'_x)}{\tan(\beta x)} + \frac{(A_xA'_x)}{\tan(\gamma x)} \;\;\Rightarrow\;\; (A_xA'_x) = \frac{\sin(\beta x)\sin(\gamma x)}{\sin((\beta+\gamma)x)}a. \tag{3}$$

From the right equation of (3) and from each of (2) we find the two expressions

$$(BA'_x) = \frac{\sin(\gamma x)\cos(\beta x)}{\sin((\beta+\gamma)x)}a, \;\; (A'_xC) = \frac{\sin(\beta x)\cos(\gamma x)}{\sin((\beta+\gamma)x)}a. \tag{4}$$

Also, from the right equation of (3) as well as from their cyclic expressions, we can readily obtain that

$$(A_xA'_x) = \tfrac{\sin(\beta x)\sin(\gamma x)}{\sin((\beta+\gamma)x)}a, \;\; (B_xB'_x) = \tfrac{\sin(\gamma x)\sin(\alpha x)}{\sin((\gamma+\alpha)x)}b, \;\; (C_xC'_x) = \tfrac{\sin(\alpha x)\sin(\beta x)}{\sin((\alpha+\beta)x)}c. \tag{5}$$

Now, from the expressions in (4) and their cyclic ones we directly have that

$$\frac{(BA'_x)}{(A'_xC)}\cdot\frac{(CB'_x)}{(B'_xA)}\cdot\frac{(AC'_x)}{(C'_xB)} = \frac{\frac{\sin(\gamma x)\cos(\beta x)}{\sin((\beta+\gamma)x)}a}{\frac{\sin(\beta x)\cos(\gamma x)}{\sin((\beta+\gamma)x)}a}\cdot\frac{\frac{\sin(\alpha x)\cos(\gamma x)}{\sin((\gamma+\alpha)x)}b}{\frac{\sin(\gamma x)\cos(\alpha x)}{\sin((\gamma+\alpha)x)}b}\cdot\frac{\frac{\sin(\beta x)\cos(\alpha x)}{\sin((\alpha+\beta)x)}c}{\frac{\sin(\alpha x)\cos(\beta x)}{\sin((\alpha+\beta)x)}c} = 1. \tag{6}$$

By virtue of (6) and the Ceva Theorem [4], AA'_x, BB'_x, CC'_x are concurrent for any $x \in (0,1]$. (*Note:* For $x=1$, the points A_1, B_1, C_1 are the three vertices A, B, C of ΔABC.) To find an analogous statement for $x=0$ we find the following limits

$$\lim_{x\to 0^+} BA'_x =: BA_0, \;\; \lim_{x\to 0^+} A'_xC =: A_0C \tag{7}$$

The above limiting expressions are obtained if we take limits of the first expression in (4) in which case we have

$$\begin{aligned}
&\lim_{x\to 0^+}(BA'_x) = \lim_{x\to 0^+}\frac{\sin(\gamma x)\cos(\beta x)}{\sin((\beta+\gamma)x)}a\\
&= \frac{\gamma\cdot\lim_{x\to 0^+}\frac{\sin(\gamma x)}{\gamma x}\cdot\lim_{x\to 0^+}\cos(\beta x)}{\beta\cdot\lim_{x\to 0^+}\frac{\sin(\beta x)}{\beta x}\cdot\lim_{x\to 0^+}\cos(\gamma x)+\gamma\cdot\lim_{x\to 0^+}\frac{\sin(\gamma x)}{\gamma x}\cdot\lim_{x\to 0^+}\cos(\beta x)}\cdot a\\
&= \frac{\gamma\cdot 1\cdot 1}{\beta\cdot 1\cdot 1+\gamma\cdot 1\cdot 1}a = \frac{\gamma}{\beta+\gamma}a =: (BA'_0) \equiv (BA_0).
\end{aligned} \tag{8}$$

Similarly, we can obtain that $\lim_{x\to 0^+}(A'_xC) = \frac{\beta}{\beta+\gamma}a =: (A'_0C) \equiv (A_0C)$ and then, cyclicly, we can have the corresponding expressions for all the other limits of the relevant line segments. Therefore, applying again the Ceva Theorem we have that

$$\frac{(BA_0)}{(A_0C)}\cdot\frac{(CB_0)}{(B_0A)}\cdot\frac{(AC_0)}{(C_0B)} = \frac{\frac{\gamma}{\beta+\gamma}a}{\frac{\beta}{\beta+\gamma}a}\cdot\frac{\frac{\alpha}{\gamma+\alpha}b}{\frac{\gamma}{\gamma+\alpha}b}\cdot\frac{\frac{\beta}{\alpha+\beta}c}{\frac{\alpha}{\alpha+\beta}c} = 1, \tag{9}$$

and so the three line segments AA_0, BB_0, CC_0 are concurrent. □

Definition 1. *For any $x \in [0,1]$ the three line segments AA'_x, BB'_x, CC'_x are called twin Hofstadter $x-$transversals of AA_x, BB_x, CC_x, respectively, while the point O_x of their concurrency is called twin Hofstadter point of H_x.*

Theorem 3. *Let O_x, $x \in [0,1]$, be the point of concurrency of AA'_x, BB'_x, CC'_x of Theorem 2 and h_{BCx}, h_{CAx}, h_{ABx} be the altitudes of the triangles ΔO_xBC, ΔO_xCA, ΔO_xAB from their common vertex O_x to BC, CA, AB, respectively.*

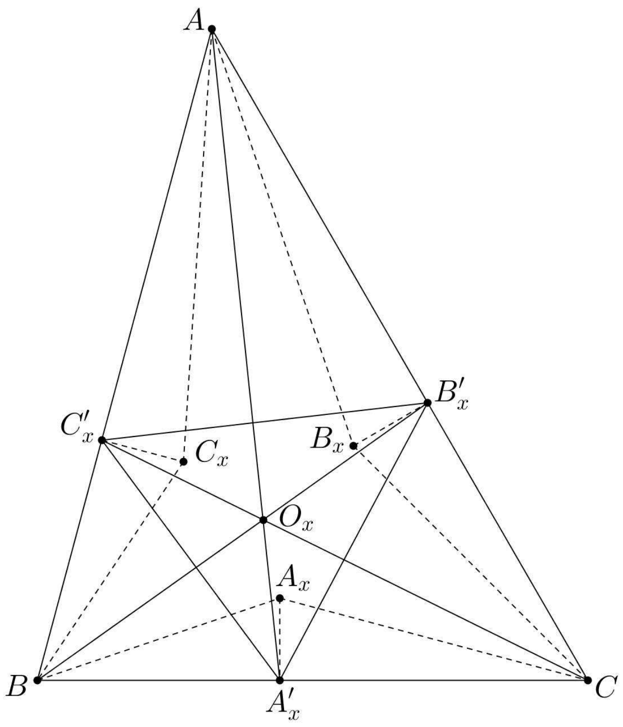

Figure 1. *Acute Case* ($\alpha = \frac{\pi}{4} < \gamma = \frac{\pi}{3} < \beta = \frac{5\pi}{6},\ x = \frac{1}{4}$)

Then, the trilinear coordinates of the twins O_x of the Hofstadter points H_x are given as follows:

$$\frac{\tan(\alpha x)}{a} : \frac{\tan(\beta x)}{b} : \frac{\tan(\gamma x)}{c} \quad \text{for } x \in (0,1),$$
$$\frac{\alpha}{a} : \frac{\beta}{b} : \frac{\gamma}{c} \quad \text{for } x = 0,$$
$$\frac{1}{\cos(\alpha)} : \frac{1}{\cos(\beta)} : \frac{1}{\cos(\gamma)} \text{ or } \frac{1}{a(b^2+c^2-a^2)} : \frac{1}{b(c^2+a^2-b^2)} : \frac{1}{c(a^2+b^2-c^2)} \quad \text{for } x = 1. \tag{10}$$

Proof. The pairs of triangles $\Delta ABA'_x$, $\Delta AA'_xC$ and $\Delta O_xBA'_x$, $\Delta O_xA'_xC$ have the same altitudes and bases the segments BA'_x and A'_xC, respectively. So, the ratios of their areas will be proportional to the lengths of their bases. In view of this we successively obtain

$$\frac{(BA'_x)}{(A'_xC)} = \frac{(ABA'_x)}{(AA'_xC)} = \frac{(O_xBA'_x)}{(O_xA'_xC)} = \frac{(ABA'_x) - (O_xBA'_x)}{(AA'_xC) - (O_xA'_xC)} = \frac{(O_xAB)}{(O_xCA)}. \tag{11}$$

By virtue of (4), the leftmost ratio above becomes $\frac{(BA'_x)}{(A'_xC)} = \frac{\tan(\gamma x)}{\tan(\beta x)}$ while the rightmost one in (11) equals $\frac{(O_xAB)}{(O_xCA)} = \frac{c\,h_{\mathrm{ABX}}}{b\,h_{\mathrm{CAX}}}$. From the two extreme ratios in (11) and their equivalent expressions just obtained we readily get $\frac{h_{\mathrm{ABX}}}{h_{\mathrm{CAX}}} = \frac{b\tan(\gamma x)}{c\tan(\beta x)}$. Considering also the other two cyclic expressions to the one just found we eventually have that

$$\frac{h_{\mathrm{ABx}}}{h_{\mathrm{CAx}}} = \frac{b\tan(\gamma x)}{c\tan(\beta x)},\ \frac{h_{\mathrm{CAx}}}{h_{\mathrm{BCx}}} = \frac{a\tan(\beta x)}{b\tan(\alpha x)},\ \frac{h_{\mathrm{BCx}}}{h_{\mathrm{ABx}}} = \frac{c\tan(\alpha x)}{a\tan(\gamma x)}. \tag{12}$$

From any two of the equalities in (12) we can obtain

$$\frac{h_{\mathrm{BCx}}}{\frac{\tan(\alpha x)}{a}} = \frac{h_{\mathrm{CAx}}}{\frac{\tan(\beta x)}{b}} = \frac{h_{\mathrm{ABx}}}{\frac{\tan(\gamma x)}{c}} \tag{13}$$

from which the first trilinear coordinates of $O_x\ \forall\, x \in (0,1)$ in (10) are immediately obtained. Note, however, that for $x = 0$ one may use the limiting expression in relation (8), as $x \to 0^+$, as well as its cyclic ones, to obtain the second trilinear coordinates for O_0 in (8). For $x = 1$ we begin with the denominator of h_{BCx} which is then $\frac{\sin(\alpha)}{a\cos(\alpha)} = \frac{1}{2R\cos(\alpha)}$, where R is the radius of the circumscribed circle to ΔABC. From the last expression and its cyclic ones we readily obtain the first set of relations in (8). To find expressions in terms of the sides of ΔABC we use the cosine formula $a^2 = b^2 + c^2 - 2bc\cos(\alpha)$ and its cyclic ones. Solving for $\cos(\alpha)$ we have $\cos(\alpha) = \frac{b^2+c^2-a^2}{2bc}$ from which the second set of expressions for $x = 1$ in (8) are produced. □

Remark 1. It should be pointed out that the twin point O_0 and the Hofstadter point H_0 coincide, while the twin point O_1 of H_1 is the orthocenter of the triangle ΔABC.

In our analysis so far it has been assumed that the given triangle is acute. If ΔABC is a right triangle, with $\beta = \frac{\pi}{2}$, or an obtuse triangle, $\beta \in \left(\frac{\pi}{2}, \pi\right)$, then some of the formulas of Theorems 2 and 3 and whatever is derived from them change.

(1) Right triangle:
- For $x \in [0,1)$ all formulas obtained in the proofs of Theorems 2 and 3 remain the same. However, as $x \to 1^-$ then

$$\lim_{x\to 1^-}(\beta x) = \frac{\pi}{2},\ \lim_{x\to 1^-}\sin(\beta x) = 1,\ \lim_{x\to 1^-}\cos(\beta x) = 0,\ \lim_{x\to 1^-}\tan(\beta x) = +\infty.$$

  So, formulas (2) and the first of (3) do not have any meaning as they stand. However, we observe that if in (3) we take limits as $x \to 1^-$ we very easily find that $(A_xA'_x) = \frac{\sin(\gamma)}{\sin(\alpha)}a = c$. Similarly, if we take limits in (4) as $x \to 1^-$ we obtain

$$\lim_{x\to 1^-}(BA'_x) = 0 \iff A'_x \equiv B,\ \lim_{x\to 1^-}(A'_xC) = (BC) = a.$$

- The applications of Ceva's Theorem in (6) and (9) remain unchanged for any $x \in [0,1)$. However for $x = 1$, when $\cos(\alpha)$ becomes zero, first we simplify by $\cos(\beta x)$ as $x \to 1^-$ and then we take limits as $x \to 1^-$; hence the end result is the same as before. Note that the point of concurrency is the vertex B, which is the orthocenter of the triangle, as was expected.

- We also have $\lim_{x\to 0^+}(BA'_x) = \lim_{x\to 0^+}\tan(\gamma x)a = 0$ meaning that $A'_x \equiv B$ something which was expected since for $\lim_{x\to 0} A'_x \equiv A_x \equiv B$ as was proved before.
- Due to the fact that $\lim_{x\to 1^-}\tan(\beta x) = +\infty$ it implies that $h_{\mathrm{BC1}} = h_{\mathrm{AB1}} = 0$. However, from the last equality in (13) we have that

$$\lim_{x\to 1^-}\frac{h_{\mathrm{BCx}}}{h_{\mathrm{ABx}}} = \frac{c\tan(\alpha)}{a\tan(\gamma)} = \frac{c\sin(\alpha)\cos(\gamma)}{a\sin(\gamma)\cos(\alpha)} = \frac{2R\cos(\gamma)}{2R\cos(\alpha)} = \frac{\frac{a}{b}}{\frac{c}{b}} = \frac{a}{c}$$

  meaning that the asymptotic convergence rate of the ratio of the altitudes from O_x to the sides BC and AB, $\frac{h_{\mathrm{BCx}}}{h_{\mathrm{ABx}}}$, as $x \to 1^-$ is $\frac{a}{c}$.
- As $x \to 1^-$, the first set of trilinear coordinates can be written as

$$\frac{\sin(\alpha x)}{a}\cos(\beta x)\cos(\gamma x) : \frac{\sin(\beta x)}{b}\cos(\gamma x)\cos(\alpha x) : \frac{\sin(\gamma x)}{c}\cos(\alpha x)\cos(\beta x).$$

  Then, taking limits as $x \to 1^-$, we observe that the two extreme coordinates become 0, due to $\lim_{x\to 1^-}\cos(\beta x) = \cos(\frac{\pi}{2}) = 0$, while the one in the middle becomes $\frac{1}{2R}\cos(\alpha)\cos(\gamma) = \frac{ca}{b^3}$. Multiplying all three coordinates found by b^2 we have that the trilinear coordinates in the third set of relations (10) become

$$0 : \frac{ac}{b} : 0 \quad \text{for } x = 1. \tag{14}$$

  Note that these trilinear coordinates give also the distances of the point $O_1 \equiv B$ from the three sides a, b, c of the right triangle ΔABC, respectively.

(2) Obtuse triangle:

- Recall that $\beta \in \left(\frac{\pi}{2}, \pi\right)$. Since $\alpha x < \gamma x < \beta x$, then for $\beta x \in (0, \frac{\pi}{2}) \Leftrightarrow x \in (0, \frac{\pi}{2\beta})$, everything will be the same as in the acute case.
- For $\beta x = \frac{\pi}{2} \Leftrightarrow x = \frac{\pi}{2\beta}$ the points $A_{\frac{\pi}{2\beta}}$ and $C_{\frac{\pi}{2\beta}}$ will be on the straight semi-lines $\mathcal{E}_{\mathrm{BC1}}$ and $\mathcal{E}_{\mathrm{BA2}}$ which will be perpendicular to the sides BC and AB, respectively. This means that the two points $A'_{\frac{\pi}{2\beta}}$ and $C'_{\frac{\pi}{2\beta}}$ will coincide with the vertex B. So, the three straight-line segments $AA'_{\frac{\pi}{2\beta}}, BB'_{\frac{\pi}{2\beta}}, CC'_{\frac{\pi}{2\beta}}$ will have as a point of concurrency the twin point $O_{\frac{\pi}{2\beta}}$ of the Hofstadter point $H_{\frac{\pi}{2\beta}}$. The various elements associated with $O_{\frac{\pi}{2\beta}}$ will be found as the limiting ones of the acute case as $x \to \left(\frac{\pi}{2\beta}\right)^-$.
- For $x \in \left(\frac{\pi}{2\beta}, 1\right)$, the angles $\beta x = \widehat{CBA_{\frac{\pi}{2\beta}}} = \widehat{ABC_{\frac{\pi}{2\beta}}} \in \left(\frac{\pi}{2}, \pi\right)$ will be obtuse meaning that the projections $A'_{\frac{\pi}{2\beta}}$ and $C'_{\frac{\pi}{2\beta}}$ of $A_{\frac{\pi}{2\beta}}$ and $C_{\frac{\pi}{2\beta}}$ will lie onto the extensions of CB and AB, respectively, and so strictly outside the triangle ΔABC. Then, all the expressions for $(BA'_x), \tan(\beta x), \cos(\beta x)$ are exactly the same as the ones in the acute case except that now all of them are negative.

The point O_x of concurrency of AA'_x, BB'_x, CC'_x will lie strictly outside the triangle and specifically in the vertically opposite angle of β.

- For $x = 1$ the various elements of interest are found from the corresponding ones in the previous item as $x \to 1^-$ and will be the orthocenter O_1 of the triangle ΔABC.

Remark 2. Summarizing, in the obtuse case there are three situations that one should have in mind. If $\beta x < \frac{\pi}{2}\,(x < \frac{\pi}{2\beta})$, then, all three points A'_x, B'_x, C'_x will lie strictly within the corresponding sides of the triangle ΔABC and Figure 2 will look like Figure 1. If $\beta x = \frac{\pi}{2}\,(x = \frac{\pi}{2\beta})$, then the points A'_x, C'_x and O_x will coincide with the vertex B. If $\pi > \beta x > \frac{\pi}{2}\,(\frac{\pi}{\beta} > x > \frac{\pi}{2\beta})$, then the points A'_x, C'_x will lie on the extensions of the sides CB and AB, respectively, while O_x will lie in the vertically opposite angle of β of the triangle ΔABC as all these are depicted in Figure 2.

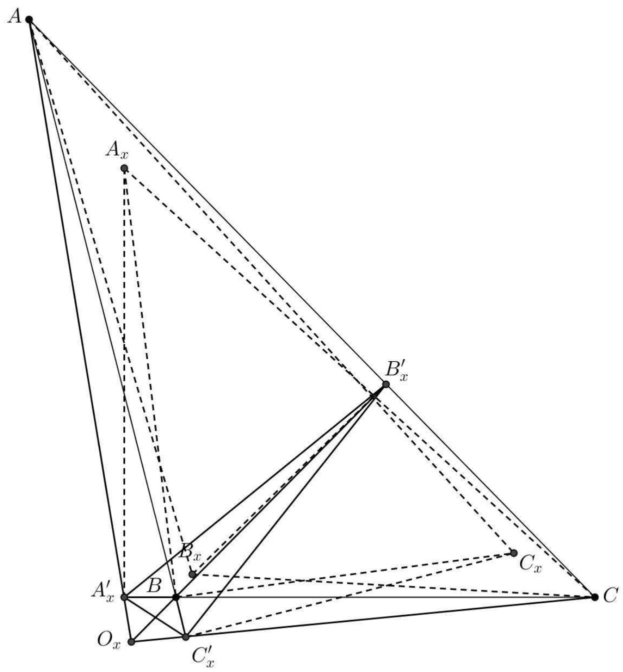

Figure 2. *Obtuse Case* ($\alpha = \frac{\pi}{6} < \gamma = \frac{\pi}{4} < \beta = \frac{5\pi}{8}$, $x = \frac{13}{14}$)

Remark 3. The exhaustive investigation of all three cases of the scalene triangle ΔABC (acute, right, obtuse) concludes our detailed analysis of the twin Hofstadter elements (triangles, transversals, points, sectrices). The cases of an isosceles or an equilateral triangle are trivial and are, therefore, omitted.

References

[1] C. Kimberling, Hofstadter points, *Nieuw Archief voor Wiskunde*, 12 (1994) 109–114.
[2] C. Maclaurin, *Sectrix of Maclaurin;* https://en.wikipedia.org/wiki/Sectrix_of_Maclairin
[3] C. O. Oakley and J. C. Baker, The Morley trisector theorem, *Amer. Math. Monthly*, 85 (1978) 737–745.
[4] A. S. Posamentier and Ch. T. Salkind, *Challenging Problems in Geometry.* Dover Publishing Co., second revised edition, 1996.

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