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Primitive Heronian Triangles With Integer Inradius and Exradii

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Abstract. It is well known that primitive Pythagorean triangles have integer inradius and exradii. We investigate the generalization to prim- itive Heronian triangles. In particular, we study the special cases of isosceles triangles and triangles with sides in arithmetic progression. We also give two families of primitive Heronian triangles, one decomposable and one indecomposable, which have integer inradii and exradii. When realized as lattice triangles, these two families have incenters and excenters at lattice points as well. Finally we pose two problems for further research.

1. Introduction

A Pythagorean triangle ABC is a right triangle with three sides a, b, $c \in \mathbb{N}$. It is primitive if $\gcd(a, b, c) = 1$.

Now suppose that ABC is a primitive Pythagorean triangle with $a^2 + b^2 = c^2$. Since $\{0, 1\}$ is the complete set of quadratic residues modulo 4, a and b must have opposite parity and c must be odd. Hence $s = \frac{1}{2}(a + b + c) \in \mathbb{N}$ and its area $T = \frac{1}{2}ab \in \mathbb{N}$. Let r, r_a, r_b, r_c be its inradius and exradii opposite A, B, C, respectively. Then r, r_a, r_b, $r_c \in \mathbb{N}$ (see Figure 1).

Conversely, if a right triangle ABC has any three of r, r_a, r_b, r_c in $\mathbb{N}$, then

$$\begin{aligned} a =&\ r + r_a = r_c - r_b \in \mathbb{N}, \\ b =&\ r + r_b = r_c - r_a \in \mathbb{N}, \\ c =&\ r_a + r_b = r_c - r \in \mathbb{N}, \end{aligned}$$

so the triangle is Pythagorean.

This motivates us to consider the more general case of Heronian triangles. A Heronian triangle ABC is a triangle with integer sides and integer area, that is, a, b, c, $T \in \mathbb{N}$. It is primitive if $\gcd(a, b, c) = 1$. Furthermore, it is *indecomposable* if h_a, h_b, h_c are not in $\mathbb{N}$, where h_a, h_b, h_c are the altitudes on a, b, c, respectively. Otherwise, it is called decomposable. Finally, we denote by I, I_a, I_b, I_c the incenter and the excenters opposite A, B, C, respectively.

2. Isosceles triangles

First, we consider the special case of isosceles triangles. For examples:

- If $(a, b, c) = (5, 5, 6)$, then $(T, r, r_a, r_b, r_c) = \left(12, \frac{3}{2}, 4, 4, 6\right)$.
- If $(a, b, c) = (13, 13, 10)$, then $(T, r, r_a, r_b, rc) = \left(60, \frac{10}{3}, 12, 12, \frac{15}{2}\right)$.

We have the following theorem.

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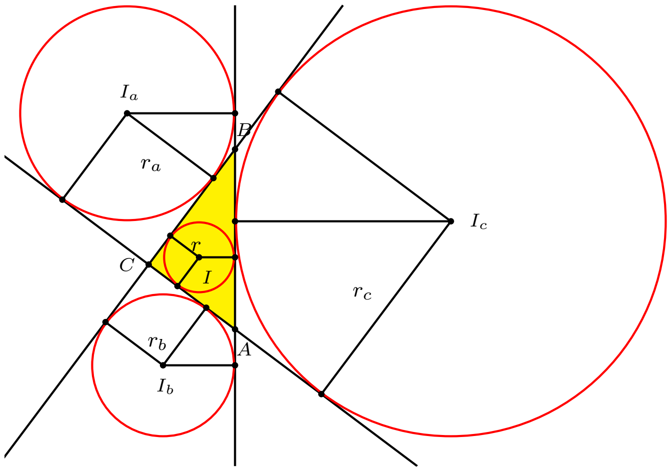

Figure 1. A right triangle ABC with its incircle and excircles

Theorem 1. *Suppose that ABC is a primitive Heronian triangle with $a = b$. Then $r_a = r_b \in \mathbb{N}$, but r and r_c cannot both be integers.*

Proof. Note that

$$s = a + \frac{c}{2}, \quad s - a = s - b = \frac{c}{2}, \quad s - c = a - \frac{c}{2},$$

so $T = \frac{c}{4}\sqrt{4a^2 - c^2}$, which implies that $4a^2 - c^2 = m^2$ for some $m \in \mathbb{N}$. Hence $-c^2 \equiv m^2 \pmod 4$, thus $c = 2d$ and $m = 2n$ for some $d, n \in \mathbb{N}$, with $\gcd(d, n) = 1$. Then $T = dn$ and $r_a = r_b = \frac{T}{s-a} = n$. Also,

$$r = \frac{T}{s} = \frac{dn}{a+d}, \qquad r_c = \frac{T}{s-c} = \frac{dn}{a-d}.$$

For contradiction, assume that $r, r_c \in \mathbb{N}$. Then

$$r_c - r = \frac{2d^2 n}{a^2 - d^2} = \frac{2d^2}{n} \in \mathbb{N},$$

which forces $n \in \{1, 2\}$. Since $n^2 = (a+d)(a-d)$, neither $n = 1$ nor $n = 2$ can yield $d \in \mathbb{N}$. This contradiction completes the proof. $\square$

3. Sides in arithmetic progression

For another special case, we consider triangles with sides in arithmetic progression. For examples:

- If $(a, b, c) = (13, 14, 15)$, then $(T, r, r_a, r_b, r_c) = \left(84, 4, \frac{21}{2}, 12, 14\right)$.
- If $(a, b, c) = (51, 52, 53)$, then $(T, r, r_a, r_b, r_c) = \left(1170, 15, \frac{130}{3}, 45, \frac{234}{5}\right)$.

We have the following theorem.

Theorem 2. *Suppose that ABC is a primitive Heronian triangle with $d = b - a = c - b > 0$. Then $r, r_b \in \mathbb{N}$. But except for $(a, b, c) = (3, 4, 5)$, r_a and r_c cannot both be integers.*

Proof. Note that

$$s = \frac{3b}{2}, \quad s - a = \frac{b}{2} + d, \quad s - b = \frac{b}{2}, \quad s - c = \frac{b}{2} - d,$$

so $T = \frac{b}{4}\sqrt{3(b^2 - 4d^2)}$, which implies that $b^2 - 4d^2 = 3m^2$ for some $m \in \mathbb{N}$. Hence $b^2 \equiv 3m^2 \pmod 4$, thus $b = 2e$ and $m = 2n$ for some e, $n \in \mathbb{N}$, with $\gcd(e, n) = 1$. Then $T = 3en$, $r = \frac{T}{s} = n$, and $r_b = \frac{T}{s-b} = 3n$. Also,

$$r_a = \frac{T}{s - a} = \frac{3en}{e + d}, \quad r_c = \frac{T}{s - c} = \frac{3en}{e - d}.$$

Assume that $r_a,\ r_c \in \mathbb{N}$. Then

$$r_a + r_c = \frac{6e^2 n}{e^2 - d^2} = \frac{2e^2}{n} \in \mathbb{N},$$

which forces $n \in \{1,\ 2\}$.

If $n = 1$, then $3 = 3n^2 = (e+d)(e-d)$, so $e = 2$ and $d = 1$, that is, $(a, b, c) = (3, 4, 5)$. If $n = 2$, then $12 = (e + d)(e - d)$. Since $\gcd(e, d) = 1$, we cannot have $(e + d, e - d) = (6, 2)$. The only remaining possibilities $(e + d, e - d) = (12, 1)$ or $(4, 3)$ cannot yield $d \in \mathbb{N}$. □

4. Triangles with r, r_a, r_b, $r_c \in \mathbb{N}$

It is possible for a primitive Heronian (non-Pythagorean) triangle to have all r, r_a, r_b, $r_c \in \mathbb{N}$. For example, if $(a, b, c) = (7, 15, 20)$, then $(T, r, r_a, r_b, r_c) = (42, 2, 3, 7, 42)$. Note that this triangle has $h_a = \frac{2T}{a} = 12$, so is decomposable. We show that there are infinitely many such decomposable ones.

Theorem 3. *There are infinitely many primitive and decomposable Heronian (non-Pythagorean) triangles with r, r_a, r_b, $r_c \in \mathbb{N}$.*

Proof. For $n > 1$, let

$$\begin{aligned}
a &= 4n^2, \\
b &= 4n^3 - 2n^2 + 1 = (2n + 1)(2n^2 - 2n + 1), \\
c &= 4n^3 + 2n^2 - 1 = (2n - 1)(2n^2 + 2n + 1).
\end{aligned}$$

Since b is odd and $a + b - c = 2$, the triangles are primitive for all $n \geq 1$. Also, from $c + 2 = a + b$ we get

$$c^2 - a^2 - b^2 = 2(ab - 2c - 2) = 2(ab - 2a - 2b + 2) \geq 0,$$

with equality if and only if $n = 1$. Therefore, for all $n > 1$, the triangles are obtuse and thus non-Pythagorean. Now,

$$\begin{aligned}
s &= 4n^3 + 2n^2 = 2n^2(2n+1),\\
s-a &= 4n^3 - 2n^2 = 2n^2(2n-1),\\
s-b &= 4n^2 - 1 = (2n-1)(2n+1),\\
s-c &= 1;\\
T &= 2n^2(2n-1)(2n+1);\\
h_a &= \frac{2T}{a} = (2n-1)(2n+1);\\
r &= \frac{T}{s} = 2n-1,\\
r_a &= \frac{T}{s-a} = 2n+1,\\
r_b &= \frac{T}{s-b} = 2n^2,\\
r_c &= \frac{T}{s-c} = 2n^2(2n-1)(2n+1) = T,
\end{aligned}$$

completing the proof. □

Notice that $n = 1$ yields the Pythagorean $(a, b, c) = (4, 3, 5)$ and $n = 2$ yields $(a, b, c) = (16, 25, 39)$.

This naturally leads to the question of whether there are such indecomposable triangles.

Theorem 4. *There are infinitely many primitive and indecomposable Heronian* (*non-Pythagorean*) *triangleswith* r, r_a, r_b, $r_c \in \mathbb{N}$.

Proof. For $n > 1$, let

$$\begin{aligned}
a &= 25n^2 + 5n - 5 = 5(5n^2 + n - 1),\\
b &= 25n^3 - 5n^2 - 7n + 3 = (5n+3)(5n^2 - 4n + 1),\\
c &= 25n^3 + 20n^2 - 2n - 4 = (5n-2)(5n^2 + 6n + 2).
\end{aligned}$$

Since a is odd and $a+b-c=2$, the triangles are primitive. Similarly, $c+2=a+b$ implies that $c^2-a^2-b^2>0$. Moreover,

$$\begin{aligned}
s =& \ 25n^3+20n^2-2n-3=(5n+3)(5n^2+n-1),\\
s-a =& \ 25n^3-5n^2-7n+2=(5n-2)(5n^2+n-1),\\
s-b =& \ 25n^2+5n-6=(5n-2)(5n+3),\\
s-c =& \ 1;\\
T =& \ (5n-2)(5n+3)(5n^2+n-1);\\
r =& \ \frac{T}{s}=5n-2,\\
r_a =& \ \frac{T}{s-a}=5n+3,\\
r_b =& \ \frac{T}{s-b}=5n^2+n-1,\\
r_c =& \ \frac{T}{s-c}=(5n-2)(5n+3)(5n^2+n-1)=T.
\end{aligned}$$

Finally,

$$\begin{aligned}
h_a =& \ \frac{2T}{a}=\frac{2(5n-2)(5n+3)}{5}\notin\mathbb{N},\\
h_b =& \ \frac{2T}{b}=\frac{2(5n-2)(5n^2+n-1)}{5n^2-4n+1}=10n+6-\frac{2}{5n^2-4n+1}\notin\mathbb{N},\\
h_c =& \ \frac{2T}{c}=\frac{2(5n+3)(5n^2+n-1)}{5n^2+6n+2}=10n-4+\frac{2}{5n^2+6n+2}\notin\mathbb{N}.
\end{aligned}$$

□

The beginning case of $n=2$ yields $(a,b,c)=(105,169,272)$.

5. Embedding as lattice triangles

In [1], Paul Yiu discovered and proved that all Heronian triangles are lattice triangles. That is, they can be embedded so that the coordinates of the three vertices are all integers. Now we wonder whether for some, I, I_a, I_b, I_c can be lattice points as well.

Theorem 5. *There are infinitely many primitive Heronian* (*non-Pythagorean*) *triangles, when realized as lattice triangles, have I, I_a, I_b, I_c at lattice points as well.*

Proof. (1) The decomposable family in Theorem 3 with

$$a=4n^2,\quad b=(2n+1)(2n^2-2n+1),\quad c=(2n-1)(2n^2+2n+1)$$

can be realized at

$$\begin{aligned} A =&\ (-2n(n-1)(2n+1),\ (2n-1)(2n+1)), \\ B =&\ (4n^2,\ 0), \\ C =&\ (0,\ 0). \end{aligned}$$

Then

$$\begin{aligned} I =&\ (s-c,\ r) = (1,\ 2n-1), \\ I_a =&\ (s-b,\ r_a) = ((2n-1)(2n+1),\ 2n+1), \\ I_b =&\ (a-s,\ r_b) = (-2n^2(2n-1),\ 2n^2), \\ I_c =&\ (s,\ r) = (2n^2(2n+1),\ 2n^2(2n-1)(2n+1)). \end{aligned}$$

For the beginning value of $n = 2$, the points are

$$\begin{aligned} &A = (-20,\ 15), \quad B = (16,\ 0), \quad C = (0,\ 0); \\ &I = (1,\ 3), \quad I_a = (15,\ 5), \quad I_b = (-24,\ 8), \text{ and } I_c = (40,\ 120). \end{aligned}$$

(2) The indecomposable family in Theorem 4 with

$$a = 5(5n^2+n-1), \quad b = (5n+3)(5n^2-4n+1), \quad c = (5n-2)(5n^2+6n+2)$$

can be realized at

$$\begin{aligned} A =&\ (2n(2n-1)(5n+3),\ (n-1)(3n-1)(5n+3)) \\ =&\ (2n(2n-1)r_a,\ (n-1)(3n-1)r_a), \\ B =&\ (-4(5n^2+n-1),\ -3(5n^2+n-1)) \\ =&\ (-4r_b,\ -3r_b), \\ C =&\ (0,0). \end{aligned}$$

Then

$$\begin{aligned} I =&\ \frac{aA+bB+cC}{a+b+c} = (3n-2,\ 4n+1), \\ I_a =&\ \frac{-aA+bB+cC}{-a+b+c} = ((-4n+1)r_a,\ (-3n+2)r_a), \\ I_b =&\ \frac{aA-bB+cC}{a-b+c} = ((4n-1)r_b,\ (3n-2)r_b), \\ I_c =&\ \frac{aA+bB-cC}{a+b-c} = ((3n-2)r_ar_b,\ (-4n+1)r_ar_b). \end{aligned}$$

For the beginning value of $n = 2$, the points are

$$\begin{aligned} &A = (156,\ 65), \quad B = (-84,\ -63), \quad C = (0,\ 0); \\ &I = (4,\ -7), \quad I_a = (-91,\ -52), \quad I_b = (147,\ 84), \text{ and } I_c = (1092,\ -1911). \end{aligned}$$

□

6. Further problems

The triangles given in the proofs of Theorem 3 and Theorem 4 are all obtuse, which suggests the following question.

Problem 1. Are there infinitely many acute primitive Heronian triangles with r, r_a, r_b, $r_c \in \mathbb{N}$?

On the other extreme, there are also primitive Heronian triangles with $r, r_a, r_b, r_c \notin \mathbb{N}$. For examples:

- If $(a, b, c) = (1921, 2929, 3600)$, then
$$(T, r, r_a, r_b, r_c) = \left(2808000, \frac{8640}{13}, \frac{4875}{4}, \frac{6500}{3}, \frac{22464}{5}\right).$$
- If $(a, b, c) = (2525, 2600, 2813)$, then
$$(T, r, r_a, r_b, r_c) = \left(3011652, \frac{47804}{63}, \frac{39627}{19}, \frac{81396}{37}, \frac{44289}{17}\right).$$

We propose the following problem.

Problem 2. Prove that there are infinitely many primitive Heronian triangles with r, r_a, r_b, $r_c \notin \mathbb{N}$.

Reference

[1] P. Yiu, Heronian triangles are lattice triangles, *Amer. Math. Monthly*, 108 (2001) 261–263.

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