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A Family of Triangles for which Two Specific Triangle Centers Have the Same Coordinates

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Abstract. It is well known that different points can have same ETC search numbers. We consider the family of triangles for which X_{3635} and X_{15519} have the same barycentric coordinates. This family includes triangle $(6, 9, 13)$. We give a geometrical construction and a locus point of view for this family of triangles.

1. Introduction

The two triangle centers in question are $X(3635)$ and $X(15519)$ in the Encyclopedia of Triangle Centers [2]. Randy Hutson [1] points out that they have the same ETC search number, while the barycentrics for $X(3635)$ are

$$(6a - b - c : \cdots : \cdots)$$

and those of $X(15519)$ are

$$((-a + b + c)(3a - b - c)^2 : \cdots : \cdots).$$

When we calculate the cross product of the coordinates of these two points we get an expression of the form

$$\{Q(a, b, c)(b - c),\ Q(a, b, c)(c - a),\ Q(a, b, c)(a - b)\}$$

where

$$Q(a, b, c) = 146abc + \sum_{\text{cyclic}} (3a^3 - 23a^2(b + c)). \tag{1}$$

The reason for the apparently strange behavior of $X(15519)$ and $X(3635)$ is the fact that $Q(6, 9, 13) = 0$.

The relation (1) can be written in terms of R, r, s as

$$5s^2 + 8r^2 - 80rR = 0,$$

which in turn is equivalent to

$$45GI^2 = 17r^2. \tag{2}$$

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2. Construction

This leads to an easy construction of the triangles satisfying (1). We first consider the following lemma.

Lemma 1. [1] *Let ABC be a triangle with incenter I. Call J the orthogonal projection of I on the parallel to BC through A, and M the midpoint of BC. Then AM meets IJ at J_0, the inverse of J with respect to the incircle.*

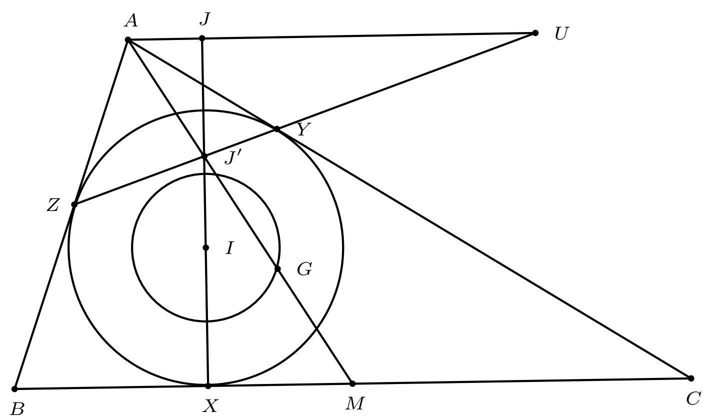

Figure 1.

Proof. Since M is the midpoint of BC and AJ is parallel to BC, AM is the harmonic conjugate of AJ with respect to AB and AC.

Let XYZ be the pedal triangle of I. If $J' = AM \cap YZ$ and $U = AJ \cap YZ$, then J' is the harmonic conjugate of U with respect to Z, Y, and therefore J' lies on the polar of U with respect to the incircle.

By the reciprocal property of pole and polar, since J' lies on the polar of U, the polar of J' goes through U, that is J is the inverse of J' with respect to the incircle. □

Now we can solve the construction problem of triangle ABC from I, G, and X (the pedal of I on BC)

Construction 2. *Given I, G, and X, construct*
(1) *D, the othogonal projection of G on the tangent to (I, IX) at X,*
(2) *L, the point on line DG such that $DG : GL = 1 : 2$,*
(3) *J, the intersection of the parallel to BC through L and line XI,*
(4) *J', inversion of J with respect to circle (I, IX),*
(5) *A, the intersection of lines GJ' and JL, and finally,*
(6) *$B,\ C$, the intersections of BC and tangents to (I, IX) from A.*

[1]This lemma was proposed by the author in the 11-th Mathematics and Computer Science National Contest "Grigore Moisil", held in Urziceni, Romania, February 2-4, 2018.

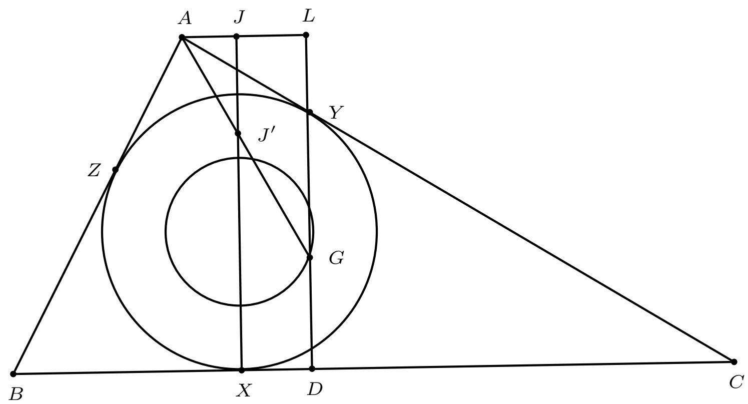

Figure 2.

Now to construct the triangles satisfying (2), we start from the circle (I, IX) and the tangent at some point X, the future line BC. If we take $r = IX$, we can perform a compass and ruler construction for the distance $g = \sqrt{\frac{17}{45}}r$ taking into account that

$$3g = \sqrt{\frac{17}{5}}r = \sqrt{\frac{4^2+1^2}{2^2+1^2}}r.$$

3. Locus

For any point G on circle (I, g) we can construct a triangle ABC satisfying (1).

What is the locus of A when G varies on circle (I, g)?

In Figure 3, X' is the reflection of X in I, and X'' the reflection of X in X'.

The curve looks like a Nichomedes conchoid with pole X' and having the parallel to BC at X'' as asymptote.

However if we take X' as the origin, and AX', $X'X''$ as axes, the cartesian equation of the curve becomes

$$(y-2r)^2x^2 = y^2\left(\sqrt{\frac{17}{5}}r + r - y\right)\left(\sqrt{\frac{17}{5}}r - r + y\right),$$

or

$$(y-2r)^2x^2 = y^2(3g + r - y)(3g - r + y),$$

while the equation of a Nichomedes conchoid is of the form

$$(y-b)^2x^2 = y^2(a + b - y)(a - b + y).$$

References

[1] R. Hutson, Advanced Plane Geometry group, message 4270, December 18, 2017; `https://groups.yahoo.com/neo/groups/AdvancedPlaneGeometry/conversations/messages/4270`.

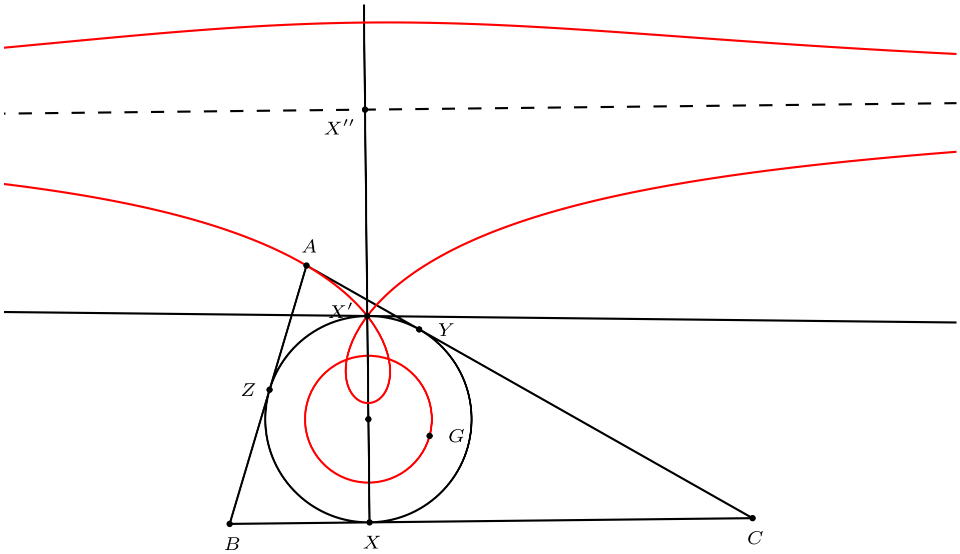

Figure 3.

[2] C. Kimberling, *Encyclopedia of Triangle Centers*, available at
http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

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