Forum Geometricorum
Volume 18 (2018) 83–86.

FORUM GEOM
ISSN 1534-1178

A New Proof of Erdős-Mordell Inequality

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Abstract. In this note we show a new proof of Erdős-Mordell inequality. The new idea is to consider three interior points to the triangle, the resulting inequality becomes Erdős-Mordell inequality when the three before mentioned points coincide.

1. Introduction

The famous Erdős-Mordell inequality states: From a point O inside a given triangle ABC the perpendiculars OP, OQ, OR are drawn to its sides. Prove that $OA+OB+OC \geq 2(OP+OQ+OR)$. Equality holds if and only if the triangle ABC is equilateral and the point O is its center. This inequality was proposed by Paul Erdős to the journal American Mathematical Monthly in 1935. Later in 1937 were published solutions by L. J. Mordell and D. F. Barrow in the same journal. A complete and extensive survey on the history of the problem can be found in [1]. There are numerous proofs in the literature to the inequality, but always considering a single point inside the triangle. So, it is natural to explore what happen for two or three interior points. More precisely, to find similar inequalities to Erdős-Mordell for more than one interior point. For two points we can find the following result in the journal Crux Mathematicorum [3].

Problem 982. (Proposed by George Tsintsifas, Thessaloniki, Greece.)
Let P and Q be interior points of triangle $A_1A_2A_3$. For $i = 1, 2, 3$, let $PA_i = x_i$, $QA_i = y_i$, and let the distances from P and Q to the side opposite A_i be p_i and q_i, respectively. Prove that

$$\sqrt{x_1y_1} + \sqrt{x_2y_2} + \sqrt{x_3y_3} \geq 2(\sqrt{p_1q_1} + \sqrt{p_2q_2} + \sqrt{p_3q_3}).$$

When $P = Q$, this reduces to the well-known Erdős-Mordell inequality.

When we consider three interior points to the triangle we only found the following result: Let ABC be a triangle and let P, Q, R be three points inside it so that $QR \perp BC$, $RP \perp CA$ and $PQ \perp AB$. Let QR meet BC at D, RP meet CA at E and PQ meet AB at F. Prove that

$$PA + QB + RC \geq PE + PF + QF + QD + RD + RE. \tag{1}$$

Publication Date: February 12, 2018. Communicating Editor: Paul Yiu.

This one is the motivation for this note since if P, Q, R coincide ($P = Q = R$), then we get Erdős-Mordell inequality. So, a solution to this problem lead to a new proof of the famous inequality. This problem was proposed as Problem 6 in a problem session from Computational Geometry course in *AwesomeMath Summer Program* 2017. During the camp, I was not aware of a solution, but in January 2018 I came back to work on this one, obtaining a satisfactory and elegant solution, my proof uses the well-known geometric lemma that is common to many different proofs of Erdős-Mordell inequality (see Lemma 1 in the next section). The lemma provides three inequalities relating the lengths of the sides of ABC and the distances from O to the vertices and to the sides. Which one has proved to be useful and today can be considered a classical result in geometric inequalities. There are different proofs for the above lemma, in [1], Claudi Alsina and Roger B. Nelsen construct a trapezoid, also in [2, p.202], the authors of the book consider the cyclic quadrilateral and orthogonal projections, obtaining the same lemma but in trigonometrical form, both are simply equivalent by Law of Sines. To tackle Problem 6 from AMSP, the last version is adequate since the inner triangle PQR has the same interior angles that triangle ABC. In this way we are relating both triangles, this task result impossible, or at least hard for the sides. Also we shall use an algebraic inequality whose proof is simple.

2. Lemmas

Lemma 1. *Given a triangle ABC and an interior point P, draw perpendiculars PY and PZ to the sides AC and AB respectively. Then we have*

$$PA \geq PY \cdot \frac{\sin C}{\sin A} + PZ \cdot \frac{\sin B}{\sin A}.$$

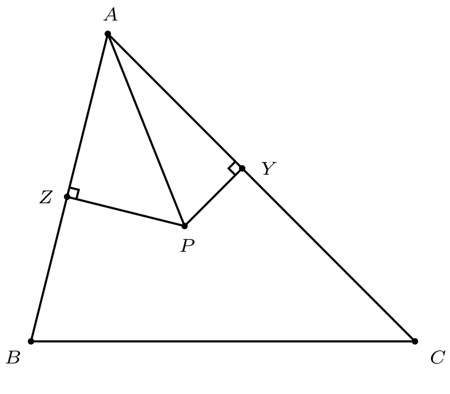

Figure 1.

Proof. See [2, page 202]. □

Lemma 2. *Let x, y, z be positive real numbers. The following inequality holds:*

$$\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z} \geq x + y + z.$$

Proof. For a positive real number α, clearly $\alpha + \frac{1}{\alpha} \geq 2$, since $(\alpha - 1)^2 \geq 0$. Now we have

$$\frac{yz}{x} + \frac{zx}{y} = z\left(\frac{y}{x} + \frac{x}{y}\right) \geq 2z.$$

Similarly,

$$\begin{aligned} \frac{zx}{y} + \frac{xy}{z} &\geq 2x, \\ \frac{yz}{x} + \frac{xy}{z} &\geq 2y. \end{aligned}$$

Summing up the three inequalities the conclusion follows. □

3. Main result

Now we are ready to prove inequality (1).

Let ABC be a triangle and let P, Q, R be three points inside it so that $QR \perp BC$, $RP \perp CA$ and $PQ \perp AB$. Let QR meet BC at D, RP meet CA at E and PQ meet AB at F. Prove that

$$PA + QB + RC \geq PE + PF + QF + QD + RD + RE.$$

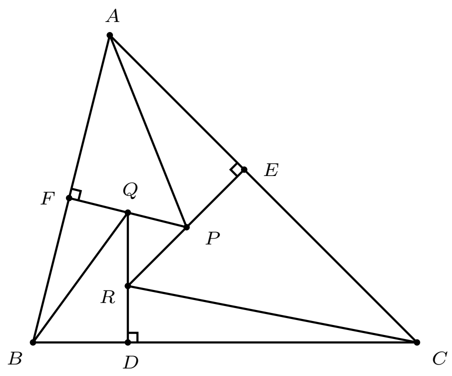

Figure 2.

By Lemma 1, the following inequalities are valid:

$$\begin{aligned} PA &\geq PE \cdot \frac{\sin C}{\sin A} + PF \cdot \frac{\sin B}{\sin A}, \\ QB &\geq QF \cdot \frac{\sin A}{\sin B} + QD \cdot \frac{\sin C}{\sin B}, \\ RC &\geq RD \cdot \frac{\sin B}{\sin C} + RE \cdot \frac{\sin A}{\sin C}. \end{aligned}$$

Quadrilateral $AFPE$ is cyclic, so $\angle FPE = 180^\circ - A$, and then $\angle QPR = A$. By analogy $\angle PQR = B$ and $\angle QRP = C$. Now, by the Law of Sines in triangle

PQR, and sum of segments, the above inequalities can be rewritten as

$$\begin{aligned} PA \geq\ & PE \cdot \frac{PQ}{QR} + PF \cdot \frac{RP}{QR} = PE \cdot \frac{PQ}{QR} + \frac{PQ \cdot RP}{QR} + \frac{QF \cdot RP}{QR}, \\ QB \geq\ & QF \cdot \frac{QR}{RP} + QD \cdot \frac{PQ}{RP} = QF \cdot \frac{QR}{RP} + \frac{QR \cdot PQ}{RP} + \frac{RD \cdot PQ}{RP}, \\ RC \geq\ & RD \cdot \frac{RP}{PQ} + RE \cdot \frac{QR}{PQ} = RD \cdot \frac{RP}{PQ} + \frac{RP \cdot QR}{PQ} + \frac{PE \cdot QR}{PQ}. \end{aligned}$$

Summing up and using Lemma 2 we obtain

$$\frac{PQ \cdot RP}{QR} + \frac{QR \cdot PQ}{RP} + \frac{RP \cdot QR}{PQ} \geq PQ + QR + RP.$$

Next, note that

$$\begin{aligned} \frac{PE \cdot PQ}{QR} + \frac{PE \cdot QR}{PQ} &\geq 2PE, \\ \frac{QF \cdot RP}{QR} + \frac{QF \cdot QR}{RP} &\geq 2QF, \\ \frac{RD \cdot PQ}{RP} + \frac{RD \cdot RP}{PQ} &\geq 2RD. \end{aligned}$$

Finally,

$$\begin{aligned} PA + QB + RC \geq\ & PQ + QR + RP + 2(PE + QF + RD) \\ =\ & PE + PF + QF + QD + RD + RE. \end{aligned}$$

References

[1] C. Alsina and R. B. Nelsen, A visual proof of the Erdős-Mordell inequality, *Forum Geom.*, 7 (2007) 99–102.

[2] T. Andreescu, O. Mushkarov, and L. Stoyanov, *Geometric Problems on Maxima and Minima*, Birkhäuser, 2006.

[3] G. Tsintsifas, Problem 982, *Crux Math.*, 10 (1984) 291.

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