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Pedals of the Poncelet Pencil and Fontené Points

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Abstract. In this essay we describe special aspects of the Poncelet pencil, pedal circles and their relation to theorems of Fontené.

1. Review of the Poncelet Pencil, [1], [2], [3]

Given triangle $\triangle ABC$ we consider the pencil of lines at the circumcenter O. For each line $\mathcal{L}$ of this pencil we apply the isogonal transformation (denoted by $'$) to obtain the resulting conic $\mathcal{K} = \mathcal{L}'$ passing through $H = O'$, the orthocenter, and thus $\mathcal{K}$ is an equilateral hyperbola. The pencil of conics $\mathcal{K}$ (as $\mathcal{L}$ varies) is called the Poncelet pencil.

It is known that the locus of centers, $Z(\mathcal{K})$, of these conics is the Euler nine point circle $\mathcal{C}_9$. The circumcircle $\mathcal{C}$ meets each conic of the Poncelet pencil in the three vertices of the triangle and the circumcircle point $W(\mathcal{K})$. The midpoint of H and $W(\mathcal{K})$ is $Z(\mathcal{K})$.

2. Pedal Circles of the Poncelet Pencil

For each point P of the plane not on the circumcircle $\mathcal{C}$ we can form the pedal triangle and then its circumcircle $\mathcal{C}(P)$. Points on $\mathcal{C}$ have pedals which lie on the Simson line.

The history of the Theorem of Griffiths, sometimes known as Fontene's second theorem, is detailed in [7], $\S 403-6$, [9], [10] and asserts the following (see also [4]).

Theorem 1. *As P varies on $\mathcal{L}$ the pedal circles $\mathcal{C}(P)$ pass through $Z(\mathcal{K})$.*

The identification of the common point is in Johnson [7], see also [2]. For the points on $\mathcal{C} \cap L$ the associated Simson lines also pass through $Z(\mathcal{K})$ as indicated in [7].

3. Pedal Circles of Isogonal Points

According to theorems proven in Honsberger, [5], p. 67, [6], p. 56, we know the following (see Figure 1).

Theorem 2. *For P and its isogonal P' the pedal triangles of $\triangle$ lie on $\mathcal{C}(P)$ with center o at the midpoint of PP'.*

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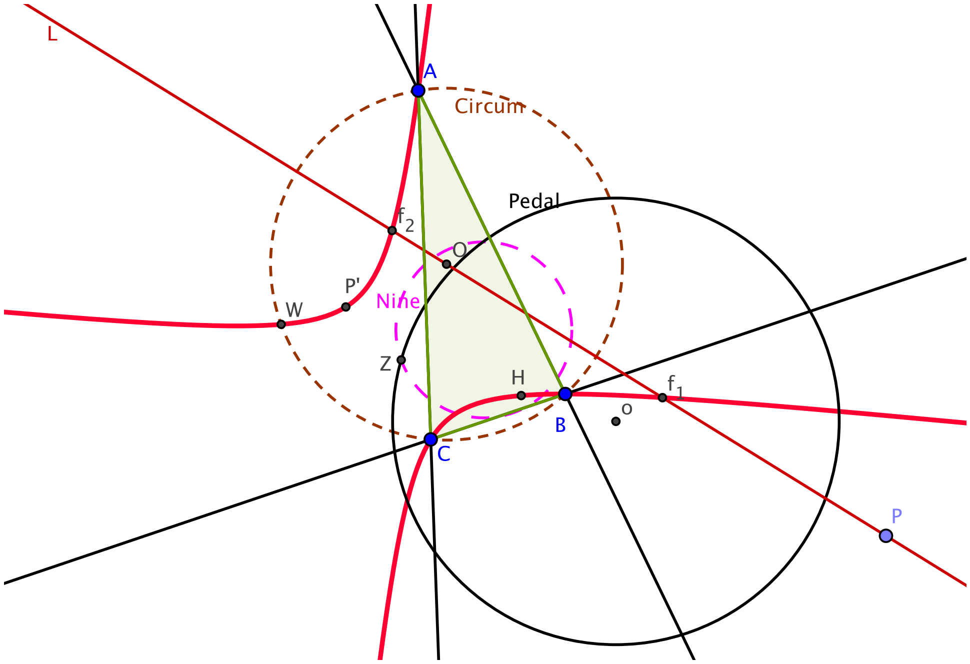

Figure 1. Pedal Circle from P

4. Fontene's Third Theorem and McCay's Cubic

Fontene's Third Theorem characterizes when the pedal circle and nine point circle, $\mathcal{C}_9$, are tangent: whenever P, O, P' are collinear.

Consider the intersections $\mathcal{L} \cap \mathcal{K}$ when $\mathcal{K}$ is irreducible. These two possible points will be called the Fontené points $\mathcal{K}$. A circle with these two Fontené points as diameter will be called the Fontené circle. By construction Fontené pairs are an isogonal pair.

Fontené's third theorem [10] can be expressed as follows (see Figure 2).

Theorem 3. *For $P \in \mathcal{L}$ the pedal circle $\mathcal{C}(P)$ and $\mathcal{C}_9$ are tangent at $Z(\mathcal{K})$ exactly when P is a Fontené point of $\mathcal{K}$.*

The next result follows immediately from the definition of the McCay cubic as the pivotal cubic determined from O and the isogonal transformation [8].

Theorem 4. *As $\mathcal{L}$ varies at O the locus of the Fontené points is the McCay cubic. The McCay cubic is self isogonal with isogonal pairs being the pairs of Fontené points of an irreducible conic of the Poncelet pencil. The reducible conics of the pencil give Fontené pairs consisting of a vertex and a point on the opposite side.*

5. Examples

5.1. *Isoceles Triangle.* In trilinear coordinates u, v, w the equation of McCay's cubic is $u(v^2 - w^2)\cos(A) + v(w^2 - u^2)\cos(B) + w(u^2 - v^2)\cos(C) = 0$. For

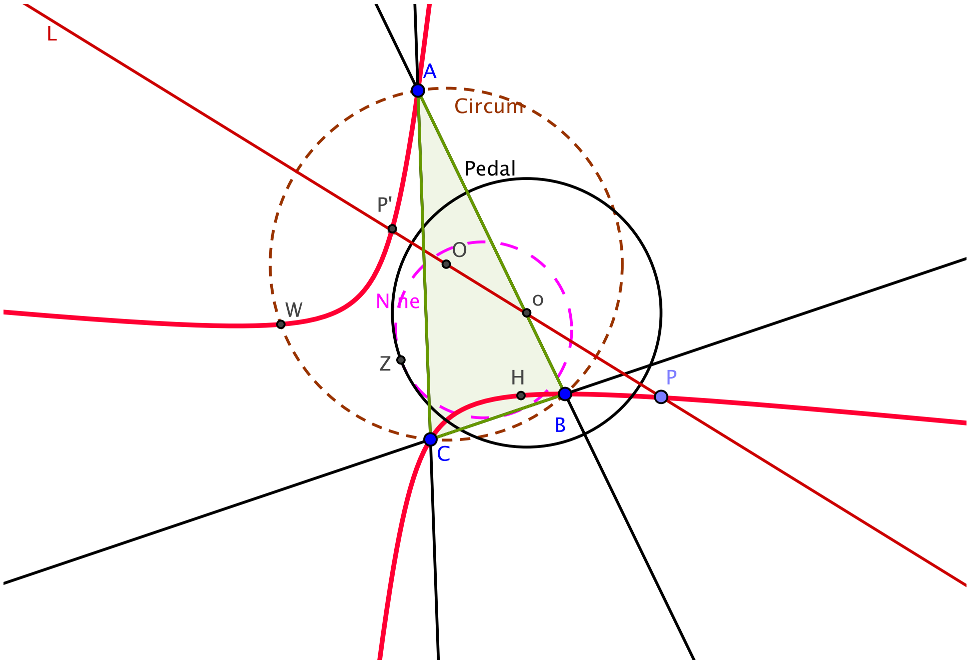

Figure 2. Tangent Pedal Circle

an isosceles triangle $A = B, u = v$ so the the equation factors up to a constant as $(u - v)(vu + w^2) = 0$. Hence McCay's cubic for an isosceles triangle consists of the perpendicular bisector $\mathcal{L}$ of the base and a hyperbola. For this line $\mathcal{L}$ there are infinitely many Fontené points on the reducible conic of the Poncelet pencil.

5.2. *Jerabek.* The Euler line $\mathcal{L}$ meets the Jerabek hyperbola $\mathcal{K} = \mathcal{L}'$ at O, H, its Fontené points. The pedal triangles from these points are the midpoint triangle and the orthic triangle. The pedal circle at these points is the Euler circle.

5.3. *Fuerbach.* $\mathcal{L} = OI$ is tangent to the Fuerbach hyperbola $\mathcal{K} = \mathcal{L}'$ at I; thus there is only one Fontené point. The pedal circle from I is the incircle tangent to the nine point circle at the Feuerbach point, the center of the Feuerbach hyperbola.

Similarly, for the other Feuerbach hyperbolas $\mathcal{K}_i = \mathcal{L}'_i$ using $\mathcal{L}_i = OI_j$ for the excenters $I_j,\ j = 1, 2, 3$ [1], we get the three ex-Feuerbach points on the nine point circle as centers of these hyperbolas. The excenter is the Fontené point.

It now follows that the McCay cubic passes through the nine points A, B, C, O, H, I, I_1, I_2, I_3 and the vertices P, Q, R of the anti-cevian triangle of the circumcenter.

5.4. *Kiepert,* $\mathcal{L} = OG'$ $, \mathcal{K} = \mathcal{L}'$.

Theorem 5. *For a non-isosceles triangle the Fontené points of the Kiepert hyperbola are complex.*

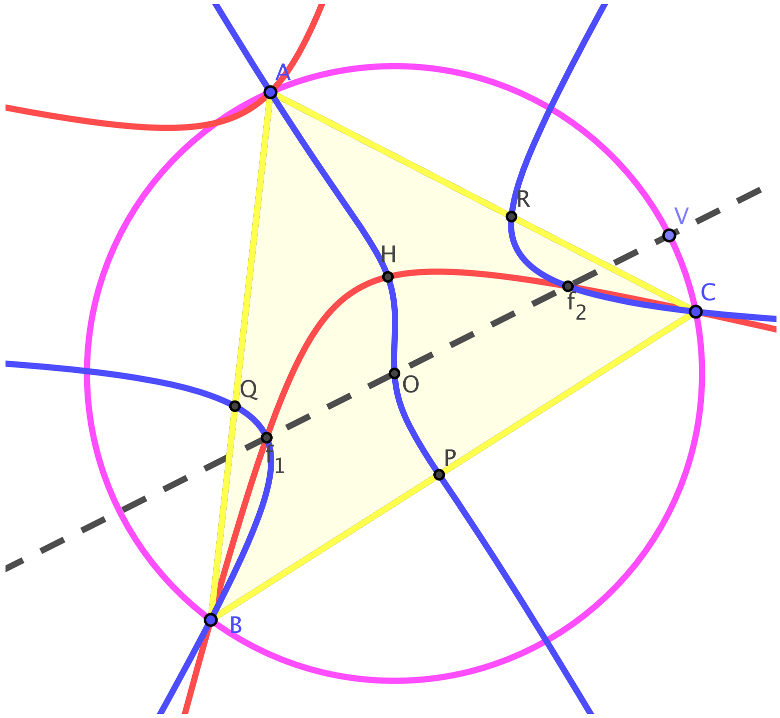

Figure 3. M'Cay Cubic

Proof. In trilinear coordinates u, v, w the McCay equation is $u(v^2 - w^2)\cos(A) + v(w^2 - u^2)\cos(B) + w(u^2 - v^2)\cos(C) = 0$ and the Kiepert's equation is $\sin(B - C)/u + \sin(C - A)/v + \sin(A - B)/w = 0$. Solving the Kiepert equation for w we can then eliminate w from the McCay equation giving a cubic equation in $x = u/v$. Since the orthocenter belongs to both curves we have a factor of $x - \cos(B)/\cos(A)$ and thus this reduces to a quadratic equation in x. Rewriting C in terms of A, B and simplifying gives the quadratic equation $x^2 - 2(\cos(A)\cos(B) + \sin(A)\sin(B)x + 1 = 0$. The discriminant is $-\sin(A - B)^2$ which is negative unless $A = B$; thus the Fontené points are complex for a non-isosceles triangle. □

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