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The Love for the Three Conics

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Abstract. Neville's theorem of the three focus-sharing conics finds a simplification and a new outreach in the context of projective geometry.

1. Introduction

In Euclidean geometry, let three points in the plane serve as three pairs of foci for three conics. The three pairs of conics define three lines through their intersections, and the three lines are concurrent, they pass a common point [4, 9, 10].

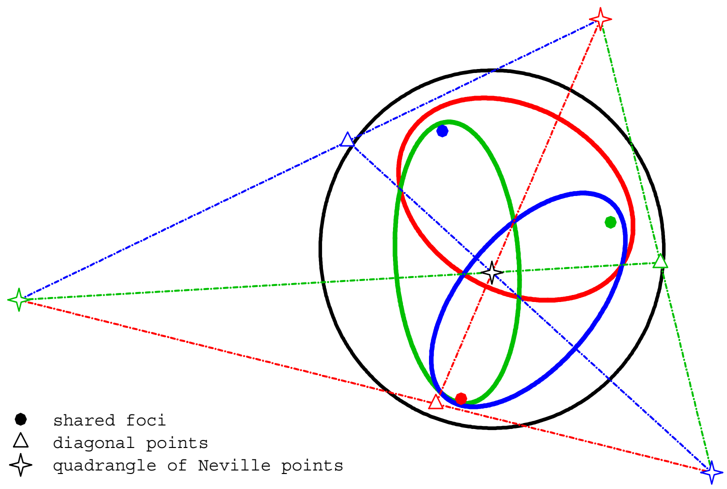

Figure 1. Three focus-sharing conics in Cayley-Klein geometries.

This theorem has found devotees and different proofs in Euclidean geometry [2, 3, 12]. It may be expanded to non-Euclidean geometries (Fig. 1) where it allows a generalising formulation in projective terms [7].

From the perspective of projective geometry, foci are defined for pairs of conics [11]. They are the vertices of the quadrilateral of common tangents and come in three pairs, which define the three diagonals of the quadrilateral [8]. Linear combination of the two conics yields the pencil of conics tangent to all four sides of the quadrilateral.

For a single conic, it is the metric-generating absolute conic of Cayley-Klein geometries which may serve as the second conic to define foci in the usual sense. This is the case in the Neville problem. Confocal conics conventionally form the pencil defined by two foci and the absolute conic in the background. Pencils exist

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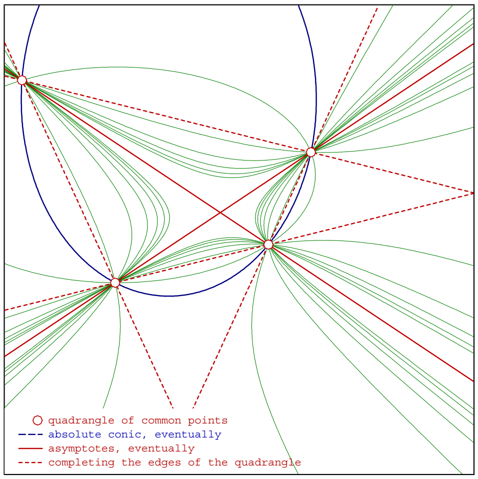
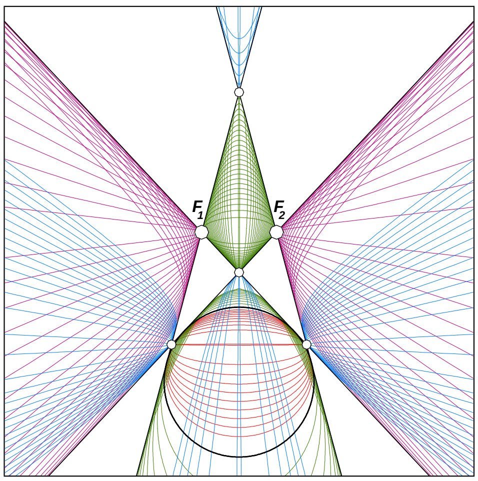


The pencil can be seen as determined by the four common points, or as generated by two conics with their four intersections. Selecting any of the conics as absolute conic (strong-line ellipse), the pencil is determined by any other conic, in particular the singular conic given by a pair of focal lines.

Figure 2. Pencil of conics through four common points.



The dual pencil can be seen as determined by the four common tangents, or as generated by two conics with their four common tangents. Selecting any of the conics as absolute conic (strong-line ellipse), the pencil is determined by any other conic, in particular the singular conic given by a pair of foci. The plane contains points where two conics of the dual pencil intersect, and points which meet none of the conics.

Figure 3. Dual pencil of conics with four common tangents.

for each pair of the three focus-sharing conics, too. It is the consideration of these pencils which admits the short and snappy proofs (see ch. 9 in [7]).

2. Pairs of conics

To explore the relation between two conics, we consider the generated pencils of conics. Figure 2 shows the pencil with the four intersection points fixed. It contains three singular elements: the three pairs of common chords. Figure 3 shows the dual pencil with the four common tangents fixed. It also contains three singular elements: the three pairs of foci. To emphasize their duality, we call the common chords also focal lines. Whichever focus we choose, it determines two directrices, which intersect in a diagonal point common to both the quadrilateral of tangents and the quadrangle of common points (Fig. 4). This diagonal point carries one of the three pairs of focal lines (i.e. common chords), see Prop. 1 below.

For Neville's problem, we start with three points (Fig. 5). For any two of them we choose a conic for which the two points are a pair of foci with respect to the absolute conic. Any two of the conics share just one focus with one crossing of directrices and one singular conic consisting of a pair of common chords. We obtain such a pair of common chords for each of the three pairs of conics. The condition that the shared foci are also foci with respect to the absolute conic imposes linear dependence, and the common chords are the edges of a quadrangle of points in which the common chords intersect. This turns out to be simple algebra, and yields the desired proofs in section 3 below.

We use homogeneous coordinates and write "$\sim$" to denote equality up to a nonzero scalar factor. Conics can be represented by pairs $[\mathbf{c}, \mathbf{C}]$ of symmetric matrices. They provide a linear map of points onto lines: $Q \mapsto \mathbf{c}Q$, the polar of Q, and its dual, a linear map of lines onto points, $g \mapsto \mathbf{C}g$, the pole of g. The pair of both maps is also called polarity. The peripheral points of a conic are the zeros of the quadratic form, $P^{\mathsf{T}}\mathbf{c}P = 0$, its tangents are the zeros of the dual, $t^{\mathsf{T}}\mathbf{C}t = 0$.

For regular conics, the two matrices $\mathbf{C}$ and $\mathbf{c}$ are reciprocal, $\mathbf{Cc} \sim \mathbf{1}$. Singular conics satisfy $\mathbf{Cc} = 0 = \mathbf{cC}$. If one of $\mathbf{c}, \mathbf{C}$ has at least rank 2, the other one is (a scalar multiple of) its adjoint (or transpose cofactor) matrix.

For $\mathbf{c}$ of rank 2, we can write $\mathbf{c} = gh^{\mathsf{T}} + hg^{\mathsf{T}}$. This yields a singular conic consisting of two distinct lines g, h of peripheral points with a pencil of tangents trough the double point $g \times h$. For $\mathbf{C}$ of rank 2, we can write $\mathbf{C} = PQ^{\mathsf{T}} + QP^{\mathsf{T}}$ and find a singular conic consisting of a (double) line $P \times Q$ of peripheral points through the concurrency centers P, Q of the two pencils of tangents.

For two distinct regular conics $[\mathbf{k}_1, \mathbf{K}_1], [\mathbf{k}_2, \mathbf{K}_2]$, the pencil $\mathcal{P}[\mathbf{k}_1, \mathbf{k}_2]$ is given by the linear combinations $\{\alpha_1\mathbf{k}_1 + \alpha_2\mathbf{k}_2\}$ (Fig. 2), the dual pencil $\tilde{\mathcal{P}}[\mathbf{K}_1, \mathbf{K}_2]$ by $\{\beta_1\mathbf{K}_1 + \beta_2\mathbf{K}_2\}$ (Fig. 3).

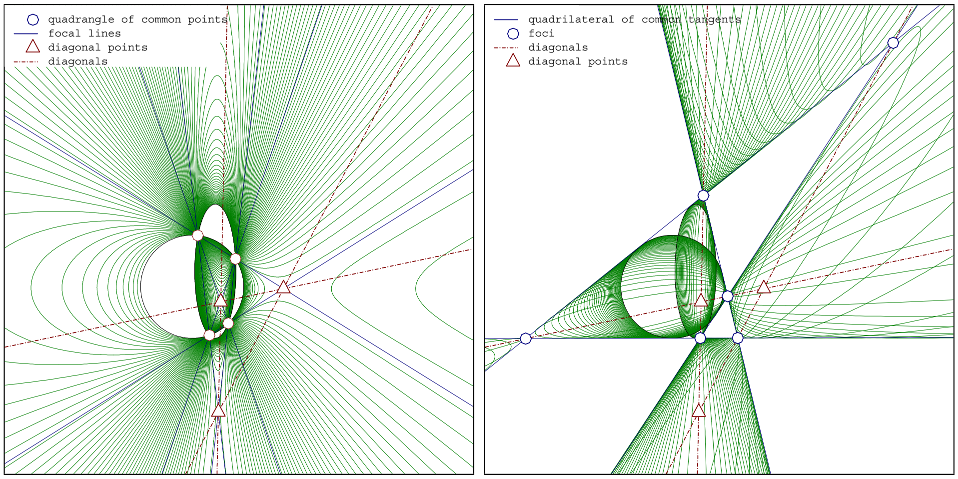
The pencil is uniquely determined by its quadrangle of common points, $\mathcal{P} = \mathcal{P}[Q_1, \ldots, Q_4]$. Its singular elements are the pairs of opposite chords, i.e. the pairs of opposite edges of the quadrangle $Q_1, \ldots, Q_4$ of common points, pairs which intersect in the diagonal points of the quadrangle. We note that for regular real conics the singular elements are real even in the case of only complex intersections of the conics.

Analogously, the dual pencil is uniquely determined by the quadrilateral of common tangents, $\tilde{\mathcal{P}} = \tilde{\mathcal{P}}[t_1, \ldots, t_4]$. Its singular elements are the diagonal lines of the quadrilateral with tangent pencils in the vertices of the quadrilateral. The six intersections of the tangents t_i, the vertices of the quadrilateral, are the common foci of the dual pencil $\tilde{\mathcal{P}}$.

Proposition 1. *For any pair of conics, the triangle given by the diagonal points of the quadrangle of common points coincides with the triangle given by the diagonal lines of the quadrilateral of common tangents. This diagonal triangle is also self-polar: Each diagonal point is the pole of the opposite diagonal line with respect to all conics of the two pencils.*

Proof. See Fig. 4 and Thm. 7.6 of [7]. □

In other words: given a pair of conics, the diagonals of the tangent quadrilateral meet in the three intersection points of opposite common chords.



Two conics have 4 common tangents and 4 common points. On the left, part of the pencil with the common points is shown, with the six chords (focal lines) and the three diagonal points. On the right, part of the pencil with the common tangents is shown, with the six intersections (foci) and the three diagonal lines. The diagonal triangles coincide.

Figure 4. Two pencils for a pair of conics with their coinciding diagonal triangles.

3. Three focus-sharing conics

In Neville's problem, three conics $\mathbf{K}_k$, $k = 1, 2, 3$ share three foci F_k, $k = 1, 2, 3$ with respect to the absolute conic $\mathbf{C}$. The pair $[F_l, F_m]$ belongs to $\mathbf{K}_k$ (k, l, m cyclically), and F_k is focus in the pairs $[\mathbf{K}_l, \mathbf{K}_m]$, $[\mathbf{K}_l, \mathbf{C}]$, and $[\mathbf{K}_m, \mathbf{C}]$.

The pair $[\mathbf{K}_l, \mathbf{K}_m]$ itself has six foci (again in three pairs). F_k is one of them, and its adjoint partner on the diagonal passing F_k will be denoted by F_k^*. The line $d_k \sim F_k \times F_k^*$ is a diagonal line of the quadrilateral of tangents common to $\mathbf{K}_l$ and $\mathbf{K}_m$. The opposite diagonal point D_k can be represented as pole of d_k to both of the conics, $D_k \sim \mathbf{K}_l d_k \sim \mathbf{K}_m d_k$, and the four polars $\mathbf{k}_l F_k, \mathbf{k}_m F_k, \mathbf{k}_l F_k^*, \mathbf{k}_m F_k^*$ are incident with D_k.

$$D_k \sim \mathbf{K}_l d_k \sim \mathbf{K}_m d_k \sim \mathbf{k}_l F_k \times \mathbf{k}_m F_k \sim \mathbf{k}_l F_k^* \times \mathbf{k}_m F_k^* . \tag{1}$$

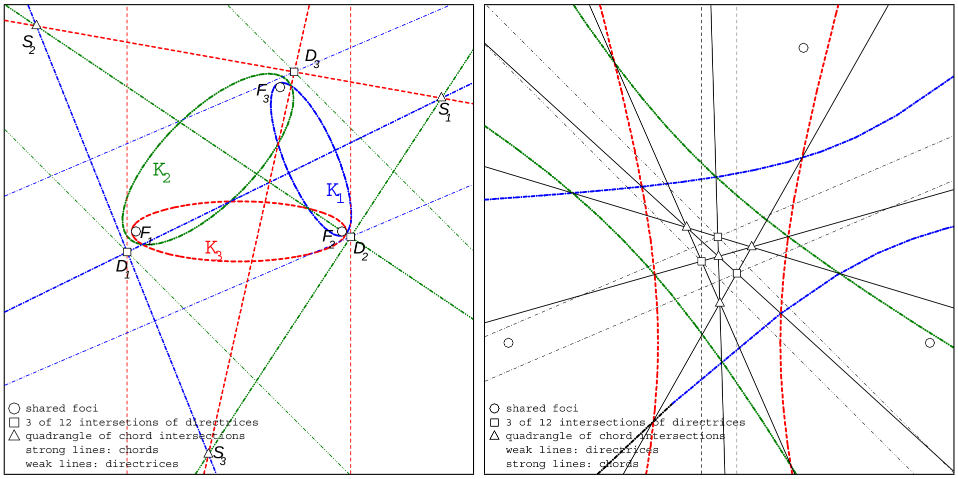
The focus F_k has a directrix (its polar) with both $\mathbf{K}_l$ and $\mathbf{K}_m$, and the intersection of the two directrices is the diagonal point D_k (Fig. 5).

We now use the identity of the diagonal triangles, Prop. 1. The diagonal point D_k is incident with two opposite chords of the pencil $\mathcal{P}(\mathbf{k}_l, \mathbf{k}_m)$. They form one of the singular elements of this pencil, $\mathbf{n}_k$. We can write

$$\mathbf{n}_k \sim (D_k^{\mathsf{T}} \mathbf{k}_l D_k)\, \mathbf{k}_m - (D_k^{\mathsf{T}} \mathbf{k}_m D_k)\, \mathbf{k}_l . \tag{2}$$

After substituting one D_k by $\mathbf{K}_l d_k$, the other by $\mathbf{K}_m d_k$, we obtain

$$\mathbf{n}_k \sim (d_k^{\mathsf{T}} \mathbf{K}_m \mathbf{k}_l \mathbf{K}_l d_k)\, \mathbf{k}_m - (d_k^{\mathsf{T}} \mathbf{K}_l \mathbf{k}_m \mathbf{K}_m d_k)\, \mathbf{k}_l \sim (d_k^{\mathsf{T}} \mathbf{K}_m d_k)\, \mathbf{k}_m - (d_k^{\mathsf{T}} \mathbf{K}_l d_k)\, \mathbf{k}_l \tag{3}$$



In the case of three ellipses, (at least) three of the common chords pass through complex intersection points. Nevertheless all six common chords are real and represent the edges of a quadrangle. In the case of three hyperbolas, we have chosen the eccentricities large enough to obtain only real intersections of the conics and real chords. For clearness, both drawings show an euclidean setting, for the general setting, see Fig. 1.

Figure 5. Three focus-sharing conics with the quadrangles of Neville centers.

This is a singular conic consisting of two lines, i.e. a pair of common chords, and a double point, the diagonal point D_k. For each pair$[\mathbf{K}_l, \mathbf{K}_m]$ of the three conics, we obtain such a singular conic consisting of that pair of opposite common chords which intersect in the diagonal point D_k.

Proposition 2. *The three singular conics $\mathbf{n}_k$ are linear dependent. Therefore, the common chords represented in these singular conics are the six sides of a quadrangle (Fig. 5).*

Proof. The conics $\mathbf{K}_m$ are elements of the dual pencils generated by $\mathbf{C}$ and the singular conics $F_k F_l^{\mathsf{T}} + F_l F_k^{\mathsf{T}}$. We fix representatives to remove the ambiguity of arbitrary nonzero scalar factors and write

$$\mathbf{K}_m = \omega_m \mathbf{C} + \lambda_m (F_k F_l^{\mathsf{T}} + F_l F_k^{\mathsf{T}}). \tag{4}$$

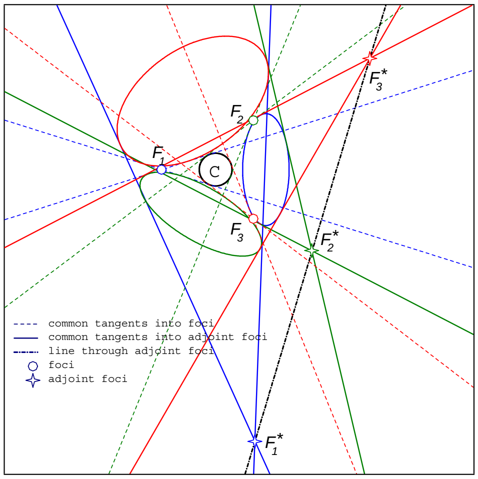
Since the diagonal d_m is incident with F_m, we obtain

$$(d_k^{\mathsf{T}} \mathbf{K}_m d_k) = \omega_m (d_k^{\mathsf{T}} \mathbf{C} d_k). \tag{5}$$

Equation (3) now yields $\mathbf{n}_{lm} = \omega_m \mathbf{k}_m - \omega_l \mathbf{k}_l$ (again choosing a suitable representative to remove the nonzero coefficient $d_k^{\mathsf{T}} \mathbf{C} d_k$) and linear dependency $\mathbf{n}_{12} + \mathbf{n}_{23} + \mathbf{n}_{31} = 0$ emerges. □

Starting with the absolute conic $\mathbf{C}$, we can reformulate Prop.2.

Corollary 3. *Given a conic with three pairs of tangents defining three intersections. Given one representative of each dual pencil generated by two of the three pairs of tangents. Then, the three intersections have two polars each (w.r.t. the*



The figure shows three conics $\mathbf{K}_m$ outside the absolute conic $\mathbf{C}$ in order to get real lines and points. The points F_m are the shared foci. The points F^*_m are opposite to the foci F_m in the quadrilateral of tangents common to $\mathbf{K}_k$ and $\mathbf{K}_l$. They are incident on a line.

Figure 6. Three focus-sharing conics with their collinear adjoint foci.

conics of the associated dual pencils). These polars intersect in points which carry two common chords. The six common chords are edges of a quadrangle.

We turn again to the adjoint foci. A pair $[\mathbf{K}_l, \mathbf{K}_m]$ defines six foci. They yield three diagonals which define a pairing of the foci, and the $[F_k, F^*_k]$ is such a pair. The focus F_k is the intersection of two tangents common to $[\mathbf{K}_l, \mathbf{K}_m]$, and F^*_k is the intersection of the two other common tangents. The focus F_k is a focus of the pair $[\mathbf{C}, \mathbf{K}_l]$ and of the pair $[\mathbf{C}, \mathbf{K}_m]$ as well. The focus adjoint to F_k in the pair $[\mathbf{K}_l, \mathbf{K}_m]$ is F^*_k, in the pair $[\mathbf{C}, \mathbf{K}_l]$ it is F_m, and in the pair $[\mathbf{C}, \mathbf{K}_m]$ it is F_l.

The following proposition was proven by Bogdanov [3] in the Euclidean case. Its generalization in Cayley-Klein geometries also generalizes the dual theorem to the four-conics-theorem (see Fig. 17 in [5]).

Proposition 4. *The three adjoint foci of the three focus-sharing conics are collinear. The six foci F_k, F^*_k, $i = 1 \ldots 3$, are the vertices of a quadrilateral (Fig. 6).*

Proof. We refer again to the singular members of dual pencils. The singular elements of the dual pencil $\mathbf{K}_{lm} = \alpha \mathbf{K}_l + \beta \mathbf{K}_m$ are given by the diagonal points itself:

$$\mathbf{N}_k = (d_k^{\mathsf{T}} \mathbf{K}_l d_k) \mathbf{K}_m - (d_k^{\mathsf{T}} \mathbf{K}_m d_k) \mathbf{K}_l. \tag{6}$$

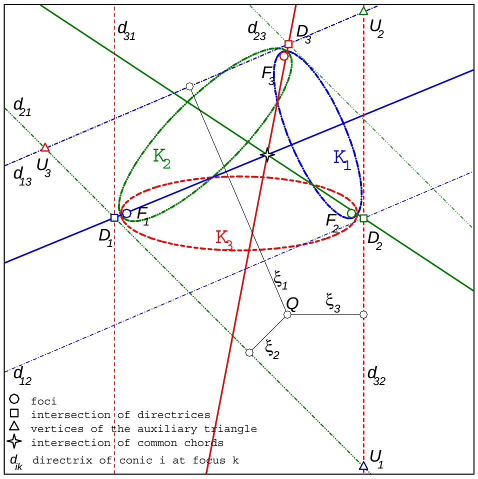
We take (5) and obtain

$$\sum_{k \in \{1,2,3\}} \omega_k (d_l^{\mathsf{T}} \mathbf{C} d_l)(d_m^{\mathsf{T}} \mathbf{C} d_m) \mathbf{N}_k = 0. \tag{7}$$

The three singular conics (all are pairs $F_k F^*_k$ of foci) are linear dependent, i.e. the foci F_k, F^*_k, $k = 1, 2, 3$ are the vertices of a complete quadrilateral, the three adjoint foci F^*_k are collinear. □

4. The limit of the Neville points

Any point can be found as a Neville point, if it is reached by the three pencils of conics generated by the three pairs of foci with the absolute conic. In particular,



Given three shared foci and three ellipses, one adds the six directrices first. We choose one of the two triangles contained in the convex hexagon (d_{13}, d_{21}, d_{32}) to establish trilinear coordinates ξ_k.
The distance of foci $(2f_i)$ is found in the distance of the directrices $(2\kappa_i = 2f_i/\varepsilon_i^2)$, so that intersections of two ellipses are given by comparing the distances from the focus (e.g. F_1) with the distances from the directrices (ξ_2 and $2\kappa_3 - \xi_3$ for the common chord of $\mathbf{K}_2$ and $\mathbf{K}_3$).

Figure 7. Three focus-sharing ellipses on the Euclidean plane.

any point in the inner part of the triangle can be the intersection of three ellipses of the set.

There is one particular point among the Neville centers. It is the limit for collapsing ellipses, for the singular members of the three dual pencils $\tilde{\mathcal{P}}[\mathbf{K}_m, \mathbf{C}]$. This limit is given by $\omega_m \to 0$ in eq. (4). We start with the Euclidean picture of Neville's proof (Fig 7). The triangle of foci is augmented with the six directrices. We identify two triangles of three directrices each. They are congruent and similar to the triangle of foci. We use distances (ξ_1, ξ_2, ξ_3) to the sides of one of them (d_{13}, d_{21}, d_{32}). After the calculation, $[\xi_1 : \xi_2 : \xi_3]$ can be reinterpreted as trilinear coordinates. We denote the eccentricity of the conic $\mathbf{K}_k$ by ε_k, the distances of the foci by $2f_k$, and the distance of the directrices by $2\kappa_k = 2f_k/\varepsilon_k^2$. The common chords of the three ellipses are

$$\varepsilon_1\xi_1 = \varepsilon_2(2\kappa_2 - \xi_2), \quad \varepsilon_2\xi_2 = \varepsilon_3(2\kappa_3 - \xi_3), \quad \varepsilon_3\xi_3 = \varepsilon_1(2\kappa_1 - \xi_1), \tag{8}$$

We obtain the intersection

$$\begin{aligned} \xi_1^S &= \kappa_1 + \varepsilon_1^{-1}(\kappa_2\varepsilon_2 - \kappa_3\varepsilon_3), \\ \xi_2^S &= \kappa_2 + \varepsilon_2^{-1}(\kappa_3\varepsilon_3 - \kappa_1\varepsilon_1), \\ \xi_3^S &= \kappa_3 + \varepsilon_3^{-1}(\kappa_1\varepsilon_1 - \kappa_2\varepsilon_2). \end{aligned} \tag{9}$$

The limit of collapsing ellipses is simply given by eccentricities equal one. In this case, the equations (8) for the chords reduce to the equations for the angular bisectors of the directrices. The common chords become the angular bisectors of the triangle $[D_1, D_2, D_3]$ approaching the focus triangle, too. The intersection of the chords approaches the incenter for eccentricities equal one.

We shall show, using the singular elements of pencils of conics, that in the general Cayley-Klein geometry, the limit of the intersection again approaches the incenter. More specifically, we shall show that the limit of the common chord of two conics sharing a focus is the angular bisector. We show the proof for the case of a regular absolute conic.

The shared foci are F_k, F_l of $[\mathbf{K}_m, \mathbf{C}]$, the diagonal points $D_m \sim \mathbf{k}_k F_m \times \mathbf{k}_l F_m \sim \mathbf{K}_k d_m \sim \mathbf{K}_l d_m$. The focal lines (i.e. the common chords) are the lines of the singular elements $\mathbf{n}_m \sim (D_m^\mathsf{T} \mathbf{k}_l D_m)\mathbf{k}_k - (D_m^\mathsf{T} \mathbf{k}_k D_m)\mathbf{k}_l$. As we are interested in the limit $\omega_m \to 0$, we fix $\lambda_m = 1$ in (4) and express the dual conics as

$$\mathbf{K}_m = \omega_m \mathbf{C} + F_k F_l^\mathsf{T} + F_l F_k^\mathsf{T}. \tag{10}$$

Proposition 5. *Given the dual conics (10). Then the the singular conics $\mathbf{n}_m$, given by (3), of the pencils $\mathcal{P}(\mathbf{k}_k, \mathbf{k}_l)$ are formed by the angular bisectors*

$$\begin{aligned} \mathbf{w}_m \quad \sim \quad & ((F_m \times F_k)^\mathsf{T} \mathbf{C} (F_m \times F_k))(F_m \times F_l)(F_m \times F_l)^\mathsf{T} \\ & -((F_m \times F_l)^\mathsf{T} \mathbf{C} (F_m \times F_l))(F_m \times F_k)(F_m \times F_k)^\mathsf{T}, \end{aligned} \tag{11}$$

i.e. $\mathbf{n}_m \sim \mathbf{w}_m$.

Proof. Step 1: $\mathbf{k}_m$ as inverse of $\mathbf{K}_m$ with suitable scaling. We normalize F_k such that $\gamma_k := F_k^\mathsf{T} \mathbf{c} F_k = \pm 1$ and write $p_k := \mathbf{c} F_k$, $\delta_m := F_k^\mathsf{T} \mathbf{c} F_l = F_k^\mathsf{T} p_l = F_l^\mathsf{T} p_k$. The coefficient ω_m in (10) is determined now. The limit of eccentricity 1 means $\omega_m \to 0$ here. The adjoint $\mathbf{k}_m$ to $\mathbf{K}_m$ is found as

$$\mathbf{k}_m = \omega_m \mathbf{K}_m^{-1} = \mathbf{c} + \frac{1}{(\mu_m^2 - \gamma_k \gamma_l)}(\gamma_l p_k p_k^\mathsf{T} + \gamma_k p_l p_l^\mathsf{T}) - \frac{\mu_m}{(\mu_m^2 - \gamma_k \gamma_l)}(p_l p_k^\mathsf{T} + p_k p_l^\mathsf{T}) \tag{12}$$

with $\mu_m = \omega_m + \delta_m$.

Step 2: Singular conics $\mathbf{n}_m$ of common chords. We use (3) and (5) from the proof of Prop. 2 to find

$$\mathbf{n}_m \sim \mathbf{k}_k - \mathbf{k}_l. \tag{13}$$

Step 3: $\mathbf{k}_m$ in the limit of eccentricity 1. We fix

$$\tilde{\mathbf{k}}_m = (F_k \times F_l)(F_k \times F_l)^\mathsf{T} \tag{14}$$

as representation of the singular conic given by the line through the foci F_k, F_l. Equation (12) yields

$$F_k^\mathsf{T} \mathbf{k}_m F_k = \frac{\gamma_k (\mu_m - \delta_m)^2}{\mu_m^2 - \gamma_k \gamma_l}, \qquad F_l^\mathsf{T} \mathbf{k}_m F_l \quad \text{the same,}$$

and finally

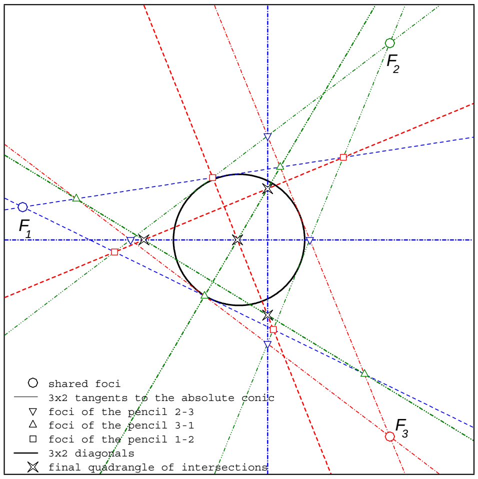
$$F_k^\mathsf{T} \mathbf{k}_m F_l = \frac{(\mu_m - \delta_m)(\mu_m \delta_m - \gamma_k \gamma_l)}{\mu_m^2 - \gamma_k \gamma_l}.$$

In the limit $\omega_m \to 0$, that is $\mu_m \to \delta_m$, these values vanish. Then both F_k and F_l are incident with the ($\mathbf{k}_m$)-polars of F_k and F_l. Consequently, the polar of any point on $F_k \times F_l$ is this line itself:

$$\lim_{\omega_m \to 0} \mathbf{k}_m \sim \tilde{\mathbf{k}}_m \tag{15}$$

Step 4: Comparing coefficients. At this point, we only have to check that the coefficients of $\mathbf{k}_k$ and $\mathbf{k}_l$ in (13) fit the coefficients in (11):

$$\lim_{\omega \to 0}(\mathbf{k}_k - \mathbf{k}_l) \overset{?}{\sim} ((F_m \times F_k)^\mathsf{T} \mathbf{C} (F_m \times F_k))\tilde{\mathbf{k}}_k - ((F_m \times F_l)^\mathsf{T} \mathbf{C} (F_m \times F_l))\tilde{\mathbf{k}}_l.$$



Each pair of foci generates a quadrilateral of tangents to the absolute conic. There are two other pairs of intersections, which are connected by the other two diagonals of the quadrilateral. These six diagonals are again edges of a quadrangle: It is the quadrangle of circumcenters of the focus triangle.

Figure 8. The quadrangle of diagonals of the set of the three pairs of foci.

Because of (15), it suffices to check one regular point. We choose $Q = \mathbf{C}(F_k \times F_l)$ and obtain

$$Q^\mathsf{T}\tilde{\mathbf{k}}_m Q = ((F_k \times F_l)^\mathsf{T} Q)^2, \qquad Q^\mathsf{T}\mathbf{k}_m Q = (F_k \times F_l)^\mathsf{T} Q$$

Therefore, indeed,

$$\begin{aligned} &\lim_{\omega\to 0}(\mathbf{k}_k - \mathbf{k}_l) \\ &= \frac{1}{(F_m \times F_l)^\mathsf{T}\mathbf{C}(F_m \times F_l)}\tilde{\mathbf{k}}_k - \frac{1}{(F_m \times F_k)^\mathsf{T}\mathbf{C}(F_m \times F_k)}\tilde{\mathbf{k}}_l \\ &\sim ((F_m \times F_k)^\mathsf{T}\mathbf{C}(F_m \times F_k))\tilde{\mathbf{k}}_k - ((F_m \times F_l)^\mathsf{T}\mathbf{C}(F_m \times F_l))\tilde{\mathbf{k}}_l. \end{aligned}$$

The singular conic of the diagonals is the singular conic of the angular bisectors, and the Neville center coincides with the incenter of the focus triangle. □

5. Trifles

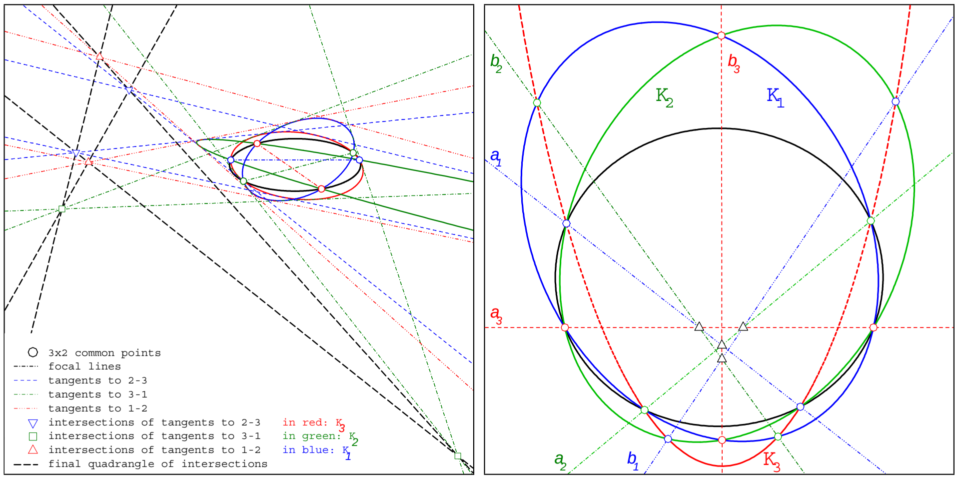
In the Euclidean setup, eq. (8), we obtain an incenter for the Neville point also in the case of equal eccentricities. The incenter is not that of the focus triangle, but of the directrix hexagon.

In the limit of infinite eccentricity, the Neville point approaches the circumcenter of the focus triangle. This circumcenter is one of the 60 Brianchon points of the tangent hexagon, which is generated by the three foci. More general: To each pair of foci belong two other pairs corresponding to the two other diagonals. These six diagonals are again edges of a quadrangle (Fig. 8).

The trilinear coordinates of the incenter of the triangle $[D_1, D_2, D_3]$ relative to the triangle $[U_1, U_2, U_3]$ in Fig. 7 are given by

$$N = [(a + b - c) : (b + c - a) : (c + a - b)]. \tag{16}$$

This seems to be a particular point [6] in the triangle $[U_1, U_2, U_3]$, but it comes in 32 different versions: The hexagon of directrices admits eight choices of directrix



Dual to the focus-sharing conics we obtain intersections of common tangents in a quadrilateral and connections of focal lines and their adjoints as edges of a quadrangle.

Figure 9. Three conics sharing focal lines.

triangles, and for each triangle a quadrangle of in- and excenters:

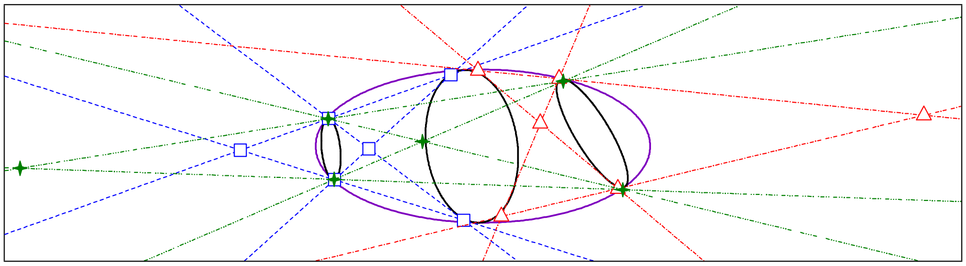
$$N = [(\pm a + s_\gamma b - s_\beta c) : (\pm b + s_\alpha c - s_\gamma a) : (\pm c + s_\beta a - s_\alpha b)] \tag{17}$$

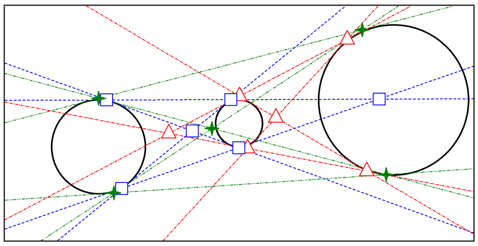
with $s_\alpha, s_\beta, s_\gamma = \pm 1$ and $s_\alpha s_\beta s_\gamma = 1$ for the quadrangle.

Since we calculate in projective spaces, the propositions of section 3 have dual counterparts. Instead of foci as intersections of tangents common with a fourth (absolute) conic, we consider the chords through intersections with this fourth (absolute) conic. We also call them focal lines to emphasize this duality. In Neville's problem, three pairs of tangents to the absolute conic generated three dual pencils from which three conics were taken. Now three pairs of points on the absolute conic generate three pencils from which three conics can be taken. In Neville's problem, each pair of the three conics had four intersections and three pairs of chords. Now each pair of three conics has four common tangents and three pairs of foci. In each pair of conics, one pair of chords intersected in the intersection of the polars of the shared focus. Now in each pair of conics, one pair of foci is collinear with the poles of the shared focal lines. In Neville's problem, the three chosen pairs of chords were the edges of a quadrangle. Now the three chosen pairs of foci are the vertices of a quadrilateral. This is the dual version of Prop. 2 (Fig. 9, left).

Each pair of three conics sharing a focal line defines an adjoint line (the partner in the pair to which the shared focal line belongs). Together with the shared focal lines we obtain six lines which are found to be the edges of a complete quadrangle. This is the dual version of Prop. 4 (Fig. 9, right), known in Euclidean notions as the four-conics theorem [5].

We conclude with a general remark on the relation between Euclidean and non-Euclidean constructions. Theorems about conics and lines only, without explicit





Three circles (in the upper part non-Euclidean, at left Euclidean) define three quadrilaterals of common tangents (differing in line style and point style). Choosing a vertex from two of these, some vertex of the third will be collinear with the two. The non-Euclidean view-point shows the connection of otherwise isolated theorems.

Figure 10. Collinearity between quadrilaterals of common tangents.

reference to symmetry and perpendiculars, can be interpreted as theorems in non-Euclidean geometry. Figure 6 is an example. One may state that if two common tangents of each pair of three conics touch a fourth conic, then the remaining common tangents of each pair intersect in three collinear points [5]. This formulation is purely Euclidean, but misses the non-Euclidean connection. In addition, Euclidean theorems about lines and circles should be expected to find simple non-Euclidean extensions by use of the non-Euclidean definitions of circles and perpendicularity. Figure 10 shows the connection between Monge's theorem, a dual three-conic theorem cited in [1, 5], and a Pascal-line construction cited in [12].

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