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A Polynomial Approach to the "Bloom" of Thymaridas and the Apollonius' Circle

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Abstract. A polynomial approach to the so-called "loom of Thymaridas" shows that it holds when it obeys a relationship that defines a circle of Apollonius.

Consider the following system of N linear equations in N unknowns:

$$\begin{aligned} x + x_1 + x_2 + \cdots + x_{N-1} &= C \\ x + x_1 &= C_1 \\ x + x_2 &= C_2 \\ &\vdots \\ x + x_{N-1} &= C_{N-1}. \end{aligned} \tag{1}$$

According to the "bloom or flower" of Thymaridas of Paros (c. 400-c. 350B.C.), the solution to this system is given by [1]:

$$x = \frac{C_1 + C_2 + \cdots + C_{N-1} - C}{N-2}. \tag{2}$$

In what follows, we show that this solution holds only if the coefficients of the quadratic polynomials whose roots enter the particular sums $(x + x_\nu)$, $\nu = 1,\ 2,\ \ldots,\ N-1$, obey a relationship that defines an Apollonius' circle as follows:

Let the left-hand side of (1) be an elementary symmetric function of degree 1 in terms of the roots of polynomial

$$\Pi(x) = a_N X^N + a_{N-1} X^{N-1} + \cdots + a_1 X + a_0$$

so that by Vieta's formulas:

$$C = -\frac{a_{N-1}}{a_N}. \tag{3}$$

Also, let any of the sums $x + x_\nu$ be the sum of the roots of quadratic polynomials $Q(X) = b_2^\nu X^2 + b_1^\nu X + b_0^\nu$ so that:

$$C_\nu = -\frac{b_1^\nu}{b_2^\nu}. \tag{4}$$

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Since x solves any two quadratic equations so that:

$$b_2^\nu x^2 + b_1^\nu x + b_0^\nu = 0 = b_2^m x^2 + b_1^m x + b_0^m$$

one obtains that:

$$(b_2^\nu - b_2^m)x^2 + (b_1^\nu - b_1^m)x + (b_0^\nu - b_0^m) = 0 \tag{5}$$

where $m = 1, 2, \ldots,\ N-1$, $m \neq \nu$. Inserting (3) and (4) in (2), and the result in (5), yields after some operations that:

$$\begin{aligned}(b_2^\nu - b_2^m)&\left[-\sum_{\nu=1}^{N-1}\frac{b_1^\nu}{b_2^\nu} - \left(-\frac{a_{N-1}}{a_N}\right)\right]^2 \\ &+ (b_1^\nu - b_1^m)\left[-\sum_{\nu=1}^{N-1}\frac{b_1^\nu}{b_2^\nu} - \left(-\frac{a_{N-1}}{a_N}\right)\right](N-2) \\ &+ (b_0^\nu - b_0^m)(N-2)^2 \\ &= 0.\end{aligned} \tag{6}$$

or, in shorthand notation:

$$B_2(A-S)^2 + B_1 S(N-2) + B_0(N-2)^2 = 0 \tag{6'}$$

where:

$$B_2 \equiv (b_2^\nu - b_2^m), \qquad B_1 \equiv (b_1^\nu - b_1^m), \qquad B_0 \equiv (b_0^\nu - b_0^m),$$
$$A \equiv \frac{a_{N-1}}{a_N} \quad \text{and} \quad S \equiv \sum_\nu \frac{b_1^\nu}{b_2^\nu}.$$

Treating ($6'$) as a quadratic equation in A:

$$B_2A^2 + [B_1(N-2) - 2B_2S]A + [B_2S^2 - B_1(N-2)S + B_0(N-2)^2] = 0$$

which has to have a unique solution and hence, a zero discriminant, one obtains after the necessary operations that:

$$A \equiv \frac{a_{N-1}}{a_N} = \frac{2B_2S - B_1(N-2)}{2B_2} = S - (N-2)\frac{B_1}{2B_2} \tag{7}$$

and

$$B_1^2 = 4B_2B_0. \tag{8}$$

When (8) is inserted in (7):

$$A \equiv \frac{a_{N-1}}{a_N} = S - (N-2)\sqrt{\frac{B_0}{B_2}} \tag{7'}$$

Given now that (2) may be rewritten in shorthand notation as follows:

$$x = \frac{A-S}{N-2} \tag{2'}$$

inserting (7) in (2'), yields:

$$x = \frac{S - (N-2)\frac{B_1}{2B_2} - S}{N-2} = -\frac{B_1}{2B_2} \equiv \frac{b_1^m - b_1^\nu}{2(b_2^\nu - b_2^m)} \tag{9}$$

or, inserting (7') in (2):

$$x = -\sqrt{\frac{B_0}{B_2}} \equiv \sqrt{\frac{b_0^m - b_0^\nu}{b_2^\nu - b_2^m}} \tag{9'}$$

The value of x has to be unique and hence, the ratio (8) or (8') between any two pairs of equations has to be the same. Geometrically, given any two points D and F along a straight line, we have a constant ratio x of varying distances from these points, distances like $(b_1^m - b_1^\nu) = PF$ and $2(b_2^\nu - b_2^m) = PD$, defining the set of points P in Figure 1, that is an Apollonius' circle centered at point O; and the set of quadratic equations that satisfy (8) or (8') is equal to the set of these points. Thymaridas' bloom holds only in this connection. And, when it does, one needs to consider the coefficients of any two quadratic equations to have a value for x.

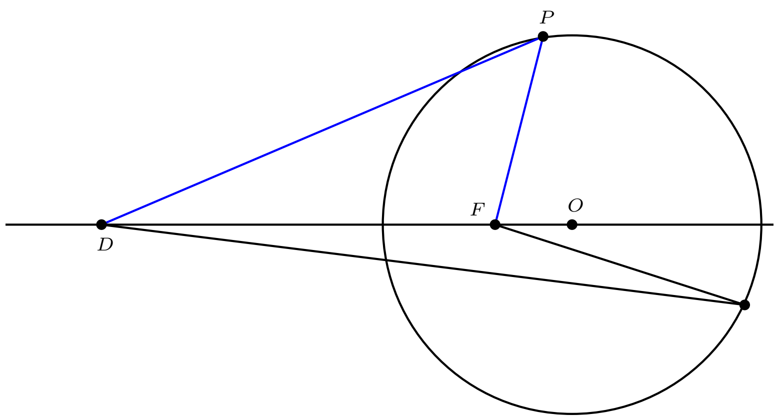

Figure 1. Thymaridas' bloom and Apollonius' circle

In sum, the bloom of Thymaridas is one way to put the algebra of the circle of Apollonius.

Reference.

[1] T. L. Heath, "The ('Bloom') of Thymaridas", in *A History of Greek Mathematics*, pp. 94–96, Clarendon Press, Oxford, 1921.

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