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Some Properties of Inversions in Alpha Plane

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Abstract. In this paper, the authors introduce inversion which is also valid in the alpha plane geometry, and give some properties such as cross ratio, harmonic conjugates with respect to inversion in the alpha plane geometry.

1. Introduction

If one wants to measure the distance between two points on a plane, then one can use frequently Euclidean distance which is defined as the length of segment between these points. Although it is the most popular distance function, it is not practical when we measure the distance which we actually move in the real world. So taxicab distance and Chinese checkers distance were introduced. Taxicab and Chinese checkers distance functions are similar to moving with a car or Chinese chess in the real world. Later, Tian [13] gave a family of metrics, α-*metric* (*alpha metric*) for $\alpha \in [0, \pi/4]$, which includes the taxicab and Chinese checkers metrics as special cases. Then, some authors developed and studied on these topics (see [5, 8, 10]). For example, Gelişgen and Kaya extended the α-distance to three and *n* dimensional spaces in [6] and [7], respectively. Afterwards, Colakoğlu [4] extended the α-metric for $\alpha \in [0, \pi/2)$. According to the latter, if $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are two points in $\mathbb{R}^2$, then for each $\alpha \in [0, \pi/2)$ and $\lambda(\alpha) = (\sec\alpha - \tan\alpha)$, the α-*distance* between *P* and *Q* is

$$d_\alpha(P,Q) = \max\{|x_1 - x_2|, |y_1 - y_2|\} + \lambda(\alpha)\min\{|x_1 - x_2|, |y_1 - y_2|\}.$$

Obviously, there are infinitely many different distance functions depending on values of α. But we suppose that values of α are initially determined and fixed unless otherwise stated.

Alpha plane geometry is a non-Euclidean geometry, and also a Minkowski geometry. Here, the linear structure is the same as the Euclidean one but distance is not *uniform* in all directions ([14]). That is, α-plane is almost the same as Euclidean plane since the points are the same, the lines are the same, and the angles are measured in the same way. Since the α-plane geometry has a different distance function it seems interesting to study the α-analog of the topics that include the concepts of distance in the Euclidean geometry.

One of the concepts which include notation of distance is an inversion. There are two kinds of transformations which are their own inverses. However, a new

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transformation also is its own inverse. This transformation is an inversion in a circle. As it has been stated in [9], this particular transformation was probably first introduced by Apollonius of Perga (225 BCE – 190 BCE). The systematic investigation of inversions began with Jakob Steiner (1796-1863) in the 1820s. During the following decades, many physicists and mathematicians independently rediscovered inversions, proving the properties that were most useful for their particular applications. For example, William Thomson used inversions to calculate the effect of a point charge on a nearby conductor made of two intersecting planes. In 1855, August F. Möbius gave the first comprehensive treatment of inversions, and Mario Pieri developed the subject axiomatically and systematically in *New Principles of the Geometry of Inversions, memoirs* I *and* II in the early 1910s, proving all of the known results as its own geometry independent of Euclidean geometry (For more detail see [9, 11]).

Since inversions have attracted attention of scientists from past to present, there are a lot of studies about them. Many scientists studied and also are studying different aspects of this concept. In [3, 12], the authors investigated the inversions with respect to the central conics in real Euclidean plane. The inversions with respect to taxicab circle was studied in detail in [1, 10].

In this paper, the authors introduce an inversion which is also valid in the alpha plane geometry, and give some properties such as cross ratio, harmonic conjugates with respect to inversion in the alpha plane geometry.

2. Preliminaries about alpha plane and some properties of alpha circular inversions

In this section, some basic concepts are briefly reviewed from [5] without proof. When one considers the d_α-metric, it is shown that the shortest path between the points P_1 and P_2 is the line segment which is parallel to a coordinate axis and a line segment making the α angle with the other coordinate axis. Thus, the shortest distance d_α between P_1 and P_2 is the sum of the Euclidean lengths of such two line segments.

Proposition 1. *Every Euclidean translation preserves distance in alpha plane. So each of them is an isometry of $\mathbb{R}^2_\alpha$.*

Proposition 2. *Let d_E and ℓ denote the Euclidean distance function and a line through the points P_1 and P_2 in the analytical plane. If ℓ has slope m; then $d_\alpha(P_1,P_2) = \frac{M}{\sqrt{1+m^2}} d_E(P_1,P_2)$ where $M = \begin{cases} 1+\lambda(\alpha)|m|, & if \ |m| \leq 1 \\ \lambda(\alpha)+|m|, & if \ |m| \geq 1 \end{cases}$.*

Proposition 2 states that d_α-distance along any line is some positive constant multiple of Euclidean distance along the same line.

Corollary 3. *Let P_1, P_2, and X be three collinear points in $\mathbb{R}^2$. Then $d_E(P_1,X) = d_E(P_2,X)$ if and only if $d_\alpha(P_1,X) = d_\alpha(P_2,X)$.*

Corollary 4. *Let P_1, P_2, and X be three distinct collinear points in $\mathbb{R}^2$. Then $d_E(P_1,X)/d_E(P_2,X) = d_\alpha(P_1,X)/d_\alpha(P_2,X)$.*

That is, the ratios of the Euclidean and d_α-distances along a line are the same. Notice that the latter corollary gives us the validity of the theorems of Menelaus and Ceva in $\mathbb{R}^2_\alpha$.

As it has been stated in [2] and [10], in the Euclidean plane an inversion in a circle of radius r is a mapping in which a point P and its image $P^{\shortmid}$ are on a ray emanating from the center O of the circle such that $d(O,P)d(O,P^{\shortmid}) = r^2$. This mapping is conformal.

Clearly if $P^{\shortmid}$ is the inverse of P, then P is the inverse of $P^{\shortmid}$. Note also that if P is in the interior of $\mathcal{C}$, $P^{\shortmid}$ is exterior to $\mathcal{C}$; and viceversa. So the interior of $\mathcal{C}$ except for $\mathcal{O}$ is mapped to the exterior and the exterior to the interior. $\mathcal{C}$ itself is left pointwise fixed. O has no image, and no point of the plane is mapped to O. However, points close to O are mapped to points far from O, and points far from O mapped to points close to O. Thus adjoining one "ideal point", or "point at infinity", to the Euclidean plane, we can include O in the domain and range of an inversion.

Now in $\mathbb{R}^2_\alpha$, the definition of inversion with respect to an α-circle can be given as follows:

Definition. Let $\mathcal{C}$ be an α-circle centered at a point O with radius r in $\mathbb{R}^2_\alpha$, and let P_∞ be the ideal point adjoining one to the alpha plane. In $\mathbb{R}^2_\alpha$ the *alpha circular inversion* with respect to $\mathcal{C}$ is the function such that

$$I_\alpha\left(O,r\right):\ \mathbb{R}^2_\alpha \cup \{P_\infty\} \to \mathbb{R}^2_\alpha \cup \{P_\infty\}$$

defined by $I_\alpha\left(O,r\right)(O) = P_\infty$, $I_\alpha\left(O,r\right)(P_\infty) = O$, and $I_\alpha\left(O,r\right)(P) = P^{\shortmid}$ for $P \neq O,\ P'$ where $P^{\shortmid}$ is on the ray $\overrightarrow{OP}$ and $d_\alpha(O,P)d_\alpha(O,P^{\shortmid}) = r^2$. The point $P^{\shortmid}$ is called the *alpha circular inverse* of P in $\mathcal{C}$; $\mathcal{C}$ is said to be *the circle of inversion*, and O is called *the center of inversion*.

The following lemma states that an inversion in a circle is a transformation of the plane that points outside the circle get mapped to points inside the circle and vice versa.

Lemma 5. *Let $\mathcal{C}$ be an alpha circle with respect to the center O in the alpha inversion $I_\alpha\left(O,r\right)$. If the point P is in the interior of $\mathcal{C}$, then the point $P^{\shortmid}$ is exterior to $\mathcal{C}$, and conversely.*

Proof. Suppose that the point P is in the interior of $\mathcal{C}$. So $d_\alpha(O,P) < r$. Since $P^{\shortmid} = I_\alpha\left(O,r\right)$, $d_\alpha\left(O,P\right) \cdot d_\alpha\left(O,P^{\shortmid}\right) = r^2$. Then

$$r^2 = d_\alpha\left(O,P\right) \cdot d_\alpha\left(O,P^{\shortmid}\right) < r \cdot d_\alpha\left(O,P^{\shortmid}\right) \text{ and } d_\alpha\left(O,P^{\shortmid}\right) > r.$$

That is, the point $P^{\shortmid}$ is in the exterior of $\mathcal{C}$. $\square$

The next proposition gives a relation getting for coordinates of $P^{\shortmid}$ in terms of coordinates of P.

Proposition 6. *Let $I_\alpha(O,r)$ be an alpha circular inversion with respect to an alpha circle $\mathcal{C}$ centered at origin and the radius r in $\mathbb{R}^2_\alpha$. If $P=(x,y)$ and $P^{\prime}=(x^{\prime},y^{\prime})$ are inverse points according to the alpha circular inversion, then*

$$x^{\prime}=\frac{r^2x}{\left(\max\{|x|,|y|\}+\lambda(\alpha)\min\{|x|,|y|\}\right)^2}\ ,$$
$$y^{\prime}=\frac{r^2y}{\left(\max\{|x|,|y|\}+\lambda(\alpha)\min\{|x|,|y|\}\right)^2}.$$

Proof. The α-circle $\mathcal{C}$ with the center origin and the radius r consists of the points which satisfies the equation $\max\{|x|,|y|\}+\lambda(\alpha)\min\{|x|,|y|\}=r$. Let $P=(x,y)$ and $P^{\prime}=(x^{\prime},y^{\prime})$ are inverse points with respect to $\mathcal{C}$. Since the points O, P and $P^{\prime}$ are collinear and the rays $\overrightarrow{OP}\ and\ \overrightarrow{OP^{\prime}}$ are same direction, $\overrightarrow{OP^{\prime}}=k\overrightarrow{OP}$ for $k\in\mathbb{R}^+$. Since $d_\alpha(O,P)\cdot d_\alpha(O,P^{\prime})=r^2$, it is obtained that

$$k=\frac{r^2}{\left(\max\{|x|,|y|\}+\lambda(\alpha)\min\{|x|,|y|\}\right)^2}.$$

Obviously the required results are obtained by substituting the value of k in $(x^{\prime},y^{\prime})=(kx,ky)$. □

The following corollary immediately is given by using the fact that all translations are isometries of alpha plane.

Corollary 7. *Let $I_\alpha(O,r)$ be an alpha circular inversion with respect to an alpha circle $\mathcal{C}$ centered at $O=(a,b)$ and the radius r. If $P=(x,y)$ is a point of $\mathbb{R}^2_\alpha$, then $P^{\prime}=(x^{\prime},y^{\prime})$ is obtained by*

$$x^{\prime}=a+\frac{r^2(x-a)}{\left(\max\{|x-a|,|y-b|\}+\lambda(\alpha)\min\{|x-a|,|y-b|\}\right)^2}\ ,$$
$$y^{\prime}=b+\frac{r^2(y-b)}{\left(\max\{|x-a|,|y-b|\}+\lambda(\alpha)\min\{|x-a|,|y-b|\}\right)^2}.$$

Now the following useful properties are well known in Euclidean plane:

i. Lines passing through the center of inversion map into themselves.
ii. Circles with center of inversion map to circles with center of inversion.
iii. Circles not passing through the center of inversion map into circles that do not pass through the center of inversion.
iv. Lines not through the center of inversion map into circles through the center of inversion and conversely.

Unfortunately all of these properties are not valid in the alpha plane. The following theorem state that whether which one of these properties are satisfied or not. Since one can easily give an example for properties which do not satisfy and one can easily prove the satisfying properties by using definition of alpha circular inversion, the next theorem is given without proof.

Theorem 8.

i. *The alpha circular inversion $I_\alpha(O,r)$ maps the lines passing through O onto themselves.*
ii. *The alpha circular inversion $I_\alpha(O,r)$ maps the alpha circles with the center O onto the alpha circles with the center O.*
iii. *The alpha circular inversion $I_\alpha(O,r)$ does not map the alpha circles not through O onto any alpha circles.*
iv. *The alpha circular inversion $I_\alpha(O,r)$ does not map the lines not containing the center of the alpha circular inversion circle onto alpha circles the center O.*
v. *The alpha circular inversion $I_\alpha(O,r)$ does not map the alpha circles containing the center of the inversion circle onto straight lines not containing O.*

3. The cross ratio and harmonic conjugates in $\mathbb{R}^2_\alpha$

The next propositions will be used to show preserving the cross ratio under alpha circular inversion.

Proposition 9. *Let $\mathcal{C}$ be an α-circle of inversion with center O and radius r, and let P, Q, and O be any three distinct collinear points in $\mathbb{R}^2_\alpha$. If P, P', and Q, Q' are pairs of inverse points, then $d_\alpha(P',\ Q') = \dfrac{r^2 d_\alpha(P,Q)}{d_\alpha(O,P)\,d_\alpha(O,Q)}$.*

Proof. Firstly suppose that O, P, Q are collinear. It is follows from definition of alpha circular inversion that $d_\alpha(O,P)\cdot d_\alpha(O,P') = d_\alpha(O,Q)\cdot d_\alpha(O,Q') = r^2$. By using Corollary 7, one can get

$$\begin{aligned} d_\alpha\left(P',Q'\right) &= \left|d_\alpha\left(O,P'\right) - d_\alpha\left(O,Q'\right)\right| \\ &= \left|\frac{r^2}{d_\alpha(O,P)} - \frac{r^2}{d_\alpha(O,Q)}\right| \\ &= \frac{r^2 d_\alpha(P,Q)}{d_\alpha(O,P)\,d_\alpha(O,Q)}. \end{aligned}$$

□

If P, Q, and O are not collinear, then the equality in Proposition 9 is not valid in $\mathbb{R}^2_\alpha$. For example, for $O=(0,0)$, $P=(1,2)$, $Q=(0,1)$ and $r=3$, the inversion $I_\alpha(O,r)$ maps P and Q into $P' = \left(\dfrac{9}{(2+\lambda(\alpha))^2}, \dfrac{18}{(2+\lambda(\alpha))^2}\right)$ and $Q' =$

$\left(0, \dfrac{9}{(2+\lambda(\alpha))^2}\right)$, respectively. One can easily see that

$$\begin{aligned} d_\alpha(P,Q) &= 1+\lambda(\alpha), \\ d_\alpha\left(P',\ Q'\right) &= \frac{9}{(2+\lambda(\alpha))^2}(1+\lambda(\alpha)), \\ d_\alpha(O,P) &= 2+\lambda(\alpha), \text{ and} \\ d_\alpha(O,Q) &= 1. \end{aligned}$$

So the equality in Proposition 9 obviously is not valid in $\mathbb{R}^2_\alpha$. But the following proposition shows that the equality in Proposition 9 is satisfied under such conditions.

Proposition 10. *Let $\mathcal{C}$ be an α-circle of inversion with center O and radius r, and let P, Q and O be any three distinct non-collinear points in $\mathbb{R}^2_\alpha$. If P, P', and Q, Q' are pairs of inverse points and P, Q lie on the lines with slope $\{0,\infty\}$ or $\{-1,1\}$ passing through the origin, then $d_\alpha(P',Q') = \dfrac{r^2 d_\alpha(P,Q)}{d_\alpha(O,P)\, d_\alpha(O,Q)}$.*

Proof. Since $P=(p,0)$ and $Q=(0,q)$ map into $P' = \left(\frac{r^2}{p},0\right),\ Q' = \left(0,\frac{r^2}{q}\right)$ or $P=(p,p)$ and $Q=(q,-q)$ map into

$$\begin{aligned} P' &= \left(\frac{r^2}{p(1+\lambda(\alpha))^2}, \frac{r^2}{p(1+\lambda(\alpha))^2}\right), \\ Q' &= \left(\frac{r^2}{q(1+\lambda(\alpha))^2}, \frac{-r^2}{q(1+\lambda(\alpha))^2}\right), \end{aligned}$$

one can easily show that

$$d_\alpha\left(P',Q'\right) = \frac{r^2 d_\alpha(P,Q)}{d_\alpha(O,P)\, d_\alpha(O,Q)}.$$

□

Let $d_\alpha[P,Q]$ denote the alpha directed distance from P to Q along a line in the alpha plane. If the ray with initial point P containing Q has the positive direction of orientation $d_\alpha[P,Q] = d_\alpha(P,Q)$, and if the ray has the opposite direction $d_\alpha[P,Q] = -d_\alpha(P,Q)$.

Now let P, Q, R and S be four distinct points on oriented line in the alpha plane. Then their *alpha cross ratio* $(PQ,RS)_\alpha$ is defined by

$$(PQ,RS)_\alpha = \frac{d_\alpha[P,R]\, d_\alpha[Q,S]}{d_\alpha[P,S]\, d_\alpha[Q,R]}.$$

Note that the alpha cross ratio is positive if both R and S are between P and Q or if neither R nor S is between P and Q, whereas the cross ratio is negative. If the pairs $\{P,Q\}$ and $\{R,S\}$ seperate each other. Also an alpha circular inversion with respect to $\mathcal{C}$ centered at origin which is different P, Q, R and S preserve the alpha cross ratio.

Theorem 11. *The alpha circular inversion preserve the alpha cross ratio.*

Proof. Suppose that P, Q, R and S be four collinear points in the alpha plane. Consider the alpha circular inversion $I_\alpha(O,r)$. Let $I_\alpha(O,r)$ map P, Q, R and S into $P^{'}$, $Q^{'}$, $R^{'}$ and $S^{'}$, respectively. First note that the alpha circular inversion preserves the seperation or non-seperation of the pairs P,Q and R,S, and also it reverses the α-directed distance from the point P to the point Q along a line *l* to α-directed distance from the point $Q^{'}$ to the point $P^{'}$. The required result follows from Proposition 9:

$$\begin{aligned}
(P^{'}Q^{'},R^{'}S^{'})_{\boldsymbol{\alpha}} &= \frac{d_\alpha(P^{'},R^{'})\,d_\alpha(Q^{'},S^{'})}{d_\alpha(P^{'},S^{'})\,d_\alpha(Q^{'},R^{'})} \\
&= \frac{\dfrac{r^2 d_\alpha(P,R)}{d_\alpha(O,P)\,d_\alpha(O,R)}\,\dfrac{r^2 d_\alpha(Q,S)}{d_\alpha(O,Q)\,d_\alpha(O,S)}}{\dfrac{r^2 d_\alpha(P,S)}{d_\alpha(O,P)\,d_\alpha(O,S)}\,\dfrac{r^2 d_\alpha(Q,R)}{d_\alpha(O,Q)\,d_\alpha(O,R)}} \\
&= \frac{d_\alpha(P,R)\,d_\alpha(Q,S)}{d_\alpha(P,S)\,d_\alpha(Q,R)} \\
&= (PQ,RS)_\alpha.
\end{aligned}$$

□

Let l be a line in $\mathbb{R}^2_\alpha$. Suppose that P, Q, R and S are four points on l. It is called that P, Q, R and S form a *harmonic set* if $(PQ,RS)_\alpha = -1$, and it is denoted by $H(PQ,RS)_\alpha$. That is, any pair R and S on l for which

$$\frac{d_\alpha[P,R]\,d_\alpha[S,Q]}{d_\alpha[P,S]\,d_\alpha[Q,R]} = -1$$

is said to divide P and Q harmonically. The points R and S are called *alpha harmonic conjugates* with respect to P and Q.

Theorem 12. *Let $\mathcal{C}$ be an alpha circle with the center O, and line segment $[PQ]$ a diameter of $\mathcal{C}$ in $\mathbb{R}^2_\alpha$. Let R and S be distinct points of the ray $\overrightarrow{OP}$, which divide the segment $[PQ]$ internally and externally. Then R and S are alpha harmonic conjugates with respect to P and Q if and only if R and S are inverse points with respect to the alpha circular inversion $I_\alpha(O,r)$.*

Proof. Let R and S are alpha harmonic conjugates with respect to *P* and *Q*. Then

$$(PQ,RS)_\alpha = -1 \Rightarrow \frac{d_\alpha[PR]\,d_\alpha[QS]}{d_\alpha[PS]\,d_\alpha[QR]} = -1.$$

Since *R* divides the line segment $[PQ]$ internally and *R* is on the ray $\overrightarrow{OQ}$,

$$d(R,Q) = r - d(O,R) \quad \text{and} \quad d(P,R) = r + d(O,R).$$

Since S divides the line segment $[PQ]$ externally and S is on the ray $\overrightarrow{OQ}$,

$$d(P,S) = \ d(O,S) + r \quad \text{and} \quad d(Q,S) = \ d(O-S) - r.$$

Hence

$$\frac{(r + d_\alpha(O,R))(d_\alpha(O,S) - r)}{(r + d_\alpha(O,S))(d_\alpha(O,R) - r)} = -1$$
$$\Rightarrow (r + d_\alpha(O,R))(d_\alpha(O,S) - r) = (r + d_\alpha(O,S))(r - d_\alpha(O,R)).$$

Simplifying the last equality, $d_\alpha(O,R) \cdot d_\alpha(O,S) = r^2$ is obtained. Therefore R and S are alpha inverse points with respect to the alpha circular inversion $I_\alpha(O,r)$.
Conversely, if R and S are alpha inverse points with respect to the alpha circular inversion $I_\alpha(O,r)$ the proof is similar. □

4. Concluding remarks

The study of inversion in the non-Euclidean planes suggest interesting and challenging problems. For example, in [1, 10], the authors investigated some properties of circular inversion in the taxicab plane, and we investigated some properties of circular inversion in the alpha plane. The obtaining results include of getting results for taxicab case since alpha distance include the taxicab and Chinese checkers distance as special cases. Moreover, we think that this topic could provoke further development by interested readers or their students.

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