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Integer Sequences and Circle Chains Inside a Hyperbola

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Abstract. In this paper we derive formulas for inscribing, inside a branch of a generic hyperbola, a chain of mutually tangent circles; moreover, we establish conditions to relate the chain of circles to certain integer sequences.

1. Introduction

Let us consider a branch of hyperbola having axis coincident with the x-axis and described by the equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad x > 0 \tag{1}$$

where a and b are arbitrary positive real numbers.

Let us inscribe inside the hyperbola a chain of circles tangent to the hyperbola itself and mutually tangent between them. See an example in Figure 1.

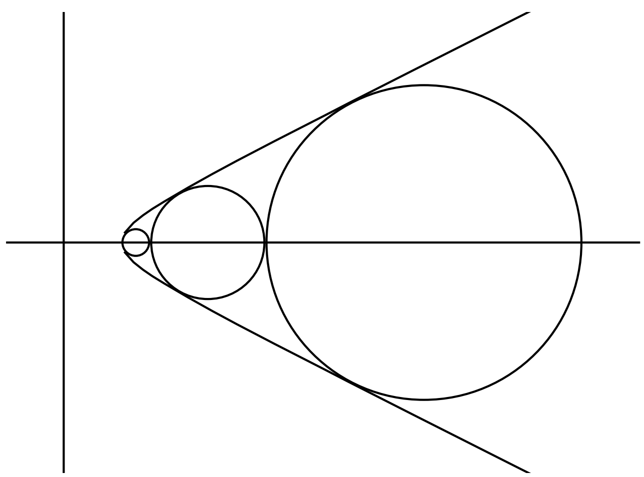

Figure 1. Circle chain inside a branch of hyperbola

In Figure 1, we have shown, for simplicity, only the right branch of the hyperbola. Nevertheless, a symmetrical circle chain can be drawn inside the left branch.

In the following, the formulas and the results that will be presented are valid for the right branch. But they can be immediately extended to the left branch by only changing x into $-x$.

By considering Figure 1, one can make the following remarks:

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- The generic n-th circle of the chain is tangent the previous $(n-1)$-th, to the $(n+1)$-th one, and to the hyperbola.
- All the centers of the circles lie on the x-axis; thus the generic n-th circle having radius r_n has center coordinates given by $(X_n,\ 0)$ with $n = 0,\ 1,\ \ldots.$
- The first circle (the smaller one identified by index 0) is tangent to the hyperbola at its vertex having coordinates $(a, 0)$; therefore, one has:

$$X_0 = a + r_0. \tag{2}$$

- Due to the mutual tangency between two consecutive circles, one can write:

$$X_n - X_{n-1} = r_n + r_{n-1}. \tag{3}$$

2. Center and radius of a generic circle of the chain

The generic n-th circle of the chain has equation:

$$(x - X_n)^2 + y^2 = r_n^2. \tag{4}$$

In order to impose the tangency condition between the n-th circle and the hyperbola, one has to start from the system:

$$\begin{cases} \frac{x^2}{a^2} - \frac{y^2}{b^2} & = 1, \\ (x - X_n)^2 + y^2 & = r_n^2. \end{cases} \tag{5}$$

From the equations in system (5):

$$(a^2 + b^2)x^2 - 2a^2 X_n x + (a^2 X_n^2 - a^2 b^2 - a^2 r_n^2) = 0. \tag{6}$$

The tangency condition between the hyperbola and the circles of the chain requests that the discriminant Δ of equation (6) is zero i.e.:

$$\frac{\Delta}{4} = a^2(a^2 b^2 + a^2 r_n^2 - b^2 X_n^2 + b^4 + b^2 r_n^2) = 0, \tag{7}$$

which yields

$$X_n^2 = \left(1 + \frac{a^2}{b^2}\right)(r_n^2 + b^2). \tag{8}$$

From (8), one can also write:

$$X_{n-1}^2 = \left(1 + \frac{a^2}{b^2}\right)(r_{n-1}^2 + b^2). \tag{9}$$

By subtracting (9) from (8) and taking into account (3) one gets:

$$X_n + X_{n-1} = \left(1 + \frac{a^2}{b^2}\right)(r_n - r_{n-1}). \tag{10}$$

Equations (3) and (10), after some algebraical steps, can be rewritten in recursive form as:

$$\begin{cases} X_n & = \left(2\frac{b^2}{a^2} + 1\right) X_{n-1} + 2\left(\frac{b^2}{a^2} + 1\right) r_{n-1}, \\ r_n & = 2\frac{b^2}{a^2} X_{n-1} + \left(2\frac{b^2}{a^2} + 1\right) r_{n-1} \end{cases} \tag{11}$$

or in matrix form

$$\begin{bmatrix} X_n \\ r_n \end{bmatrix} = \begin{bmatrix} 2\frac{b^2}{a^2}+1 & 2\frac{b^2}{a^2}+2 \\ 2\frac{b^2}{a^2} & 2\frac{b^2}{a^2}+1 \end{bmatrix} \begin{bmatrix} X_{n-1} \\ r_{n-1} \end{bmatrix} \tag{12}$$

that allows to express X_n and r_n in terms of X_0 and r_0 by means of the following relation:

$$\begin{bmatrix} X_n \\ r_n \end{bmatrix} = \begin{bmatrix} 2\frac{b^2}{a^2}+1 & 2\frac{b^2}{a^2}+2 \\ 2\frac{b^2}{a^2} & 2\frac{b^2}{a^2}+1 \end{bmatrix}^n \begin{bmatrix} X_0 \\ r_0 \end{bmatrix} \tag{13}$$

Finally, by means of (2) and (8) it possible to write an explicit expression for X_0 and r_0 in function of the hyperbola parameters a and b:

$$\begin{bmatrix} X_0 \\ r_0 \end{bmatrix} = \begin{bmatrix} \frac{b^2+a^2}{a} \\ \frac{b^2}{a} \end{bmatrix} \tag{14}$$

3. Integer sequences associated with circle chains

In this paragraph, we want to establish possible connections between the circle chains and certain integer sequences. To this aim, it is useful to introduce the new variables:

$$\widetilde{X_n} = \frac{X_n}{X_0}, \qquad \widetilde{r_n} = \frac{r_n}{r_0}$$

so that, by remembering equation (14), equation (13) becomes:

$$\begin{bmatrix} \widetilde{X_n} \\ \widetilde{r_n} \end{bmatrix} = \begin{bmatrix} 2\frac{b^2}{a^2}+1 & 2\frac{b^2}{a^2} \\ 2\frac{b^2}{a^2}+2 & 2\frac{b^2}{a^2}+1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{15}$$

From (15), one can generate two sequences $\{\widetilde{X_n}\}$ and $\{\widetilde{r_n}\}$ that both depend on the ratio b/a.

Now, one may pose the question: is it possible to find values for the ratio b/a so that $\{\widetilde{X_n}\}$ and $\{\widetilde{r_n}\}$ are integer sequences? The answer is affirmative as stated by the following theorem:

Theorem. *If the ratio b/a is given by*
(CASE A): $\frac{b}{a} = k$, $k = 1,\ 2,\ \ldots$ (16a)
or
(CASE B): $\frac{b}{a} = \frac{2k+1}{2}$, $k = 0,\ 1, \ldots,$ (16b)
then $\{\widetilde{X_n}\}$ and $\{\widetilde{r_n}\}$ are integer sequences.

Proof. (Case A) From (16a) we have that the ratio b/a is an integer and the elements of the 2×2 matrix in (15) are integers; therefore, also any generic n-th power of the matrix will be composed by only integers. Hence, from (15) one can conclude that, for any value of n, both $\widetilde{X_n}$ and $\widetilde{r_n}$ are integers.

(Case B) The 2×2 matrix in (15), that we name $[M(k)]$, in this case, becomes:

$$[M(k)] = \begin{bmatrix} \frac{4k^2+4k+5}{2} & \frac{4k^2+4k+1}{2} \\ \frac{4k^2+4k+9}{2} & \frac{4k^2+4k+5}{2} \end{bmatrix} \tag{17}$$

First of all, we demonstrate by induction that the n-th power of $[M(k)]$ can be written in the following form:

$$[M(k)]^n = \begin{bmatrix} \frac{P_{2n}(k)}{2} & \frac{(4k^2+4k+1)Q_{2n-2}(k)}{2} \\ \frac{(4k^2+4k+9)Q_{2n-2}(k)}{2} & \frac{P_{2n}(k)}{2} \end{bmatrix} \tag{18}$$

where $P_{2n}(k)$ and $Q_{2n-2}(k)$ are two polynomial functions of the integer variable k of degrees $2n$ and $2n-2$ respectively, both generating odd integers. Moreover, a further relation between them holds:

$$\frac{P_{2n}(k) + Q_{2n-2}(k)}{2} = \text{odd integer} \tag{19}$$

Equations (18) and (19) represent the inductive hypothesis $H(n)$.

For $n = 1$, one immediately notes, from (17), that the inductive hypothesis is true; in this case $P_2 = 4k^2 + 4k + 5$ and $Q_0 = 1$ and also (19) is verified.

Now we demonstrate that if $H(n)$ is true, then $H(n+1)$ too is true.

The starting point is the following relation:

$$[M(k)]^{n+1} = \begin{bmatrix} \frac{4k^2+4k+5}{2} & \frac{4k^2+4k+1}{2} \\ \frac{4k^2+4k+9}{2} & \frac{4k^2+4k+5}{2} \end{bmatrix} \begin{bmatrix} \frac{P_{2n}(k)}{2} & \frac{(4k^2+4k+1)Q_{2n-2}(k)}{2} \\ \frac{(4k^2+4k+9)Q_{2n-2}(k)}{2} & \frac{P_{2n}(k)}{2} \end{bmatrix} \tag{20}$$

From (20) and after some algebraical steps, one obtains:

$$[M(k)]^{n+1} = \begin{bmatrix} \frac{(4k^2+4k+5)P_{2n}(k)+(4k^2+4k+1)(4k^2+4k+9)Q_{2n-2}(k)}{4} & \frac{(4k^2+4k+1)\left[(4k^2+4k+5)Q_{2n-2}(k)+P_{2n}(k)\right]}{4} \\ \frac{(4k^2+4k+9)\left[(4k^2+4k+5)Q_{2n-2}(k)+P_{2n}(k)\right]}{4} & \frac{(4k^2+4k+5)P_{2n}(k)+(4k^2+4k+1)(4k^2+4k+9)Q_{2n-2}(k)}{4} \end{bmatrix} \tag{21}$$

that can be written as:

$$[M(k)]^{n+1} = \begin{bmatrix} \frac{P_{2(n+1)}(k)}{4} & \frac{(4k^2+4k+1)Q_{2(n+1)-2}(k)}{4} \\ \frac{(4k^2+4k+9)Q_{2(n+1)-2}(k)}{4} & \frac{P_{2(n+1)}(k)}{4} \end{bmatrix} \tag{22}$$

where:

$$P_{2(n+1)}(k) = \frac{(4k^2+4k+5)P_{2n}(k) + (4k^2+4k+1)(4k^2+4k+9)Q_{2n-2}(k)}{2}, \tag{23}$$

$$Q_{2(n+1)-2}(k) = \frac{(4k^2+4k+5)Q_{2n-2}(k) + P_{2n}(k)}{2}. \tag{24}$$

Furthermore, equation (23), after some algebraical steps, can be written as:

$$\begin{aligned} P_{2(n+1)}(k) = {} & 2(k^2+k+1)P_{2n}(k) + 2(4k^2+4k+1)(k^2+k+2)Q_{2n-2}(k) \\ & + 2(k^2+k)Q_{2n-2}(k) + \frac{P_{2n}(k) + Q_{2n-2}(k)}{2}. \end{aligned} \tag{25}$$

By taking into account (19), one immediately notices that (25) is an odd integer.

In an analogous way, one can write:

$$Q_{2(n+1)-2}(k) = 2(k^2+k+1)Q_{2n-2}(k) + \frac{P_{2n}(k)+Q_{2n-2}(k)}{2}. \tag{26}$$

By taking into account (26), one immediately notices that (26) is an odd integer. Finally, by adding (25) and (26), and dividing by 2, one gets:

$$\begin{aligned}\frac{P_{2(n+1)}(k)+Q_{2(n+1)-2}(k)}{2} = & \frac{P_{2n}(k)+Q_{2n-2}(k)}{2}(2k^2+2k+3) \\ & + 2(2k^4+4k^3+3k^2+9k+1)Q_{2n-2}(k)\end{aligned} \tag{27}$$

By remembering (19) one has that the first addend in (27) is an odd integer. Conversely the second addend in (27) is an even integer. Thus (27) is an odd integer.

This concludes the demonstration by induction; therefore, equation (18) is true for each $n \geq 1$.

Now, by remembering (18) and (15) , we can write:

$$\begin{bmatrix}\widetilde{X_n} \\ \widetilde{r_n}\end{bmatrix} = [M(k)]^n \begin{bmatrix}1 \\ 1\end{bmatrix} = \begin{bmatrix}\frac{P_{2n}(k)}{2} & \frac{(4k^2+4k+1)Q_{2n-2}(k)}{2} \\ \frac{(4k^2+4k+9)Q_{2n-2}(k)}{2} & \frac{P_{2n}(k)}{2}\end{bmatrix}\begin{bmatrix}1 \\ 1\end{bmatrix} \tag{28}$$

so that

$$\begin{bmatrix}\widetilde{X_n} \\ \widetilde{r_n}\end{bmatrix} = \begin{bmatrix}\frac{P_{2n}(k)+(4k^2+4k+1)Q_{2n-2}(k)}{2} \\ \frac{(4k^2+4k+9)Q_{2n-2}(k)+P_{2n}(k)}{2}\end{bmatrix} \tag{29}$$

On the right hand side of (29), both numerators are sums of two odd integers. Therefore, each of them is an even integer. Consequently both $\widetilde{X_n}$ and $\widetilde{r_n}$ are integers for every integer n. This concludes the proof. □

4. Integer sequences classified in OEIS

In the previous paragraph, we have shown that if equations (16a) or (16b) hold, then the sequences $\{\widetilde{X_n}\}$ and $\{\widetilde{r_n}\}$ are composed by integer numbers. By varying the value of the parameter b/a one can generate an infinite number of integer sequences. A certain number of them are classified in OEIS (On-Line Encyclopedia of Integer Sequences) [1]. The results, we found, are shown in Table I.

Table I: Integer sequences associated with circle chains
and classified in OEIS

Ratio b/a	$\{\widetilde{X_n}\}$	$\{\widetilde{r_n}\}$
$1/2$	$A001519$	$A002878$
1	$A001653$	$A002315$
$3/2$	$A078922$	$A097783$
2	$A007805$	$A049629$
$5/2$	$A097835$	$A097834$
3	$A097315$	$A097314$
$7/2$	$A097838$	$A097837$
4	$A078988$	$A078989$
$9/2$	$A097841$	$A097840$
5	$A097727$	$A097726$
$11/2$	$A097843$	$A097842$
6	$A097730$	$A097729$
$13/2$	$A098244$	$A097845$
7	$A097733$	$A097732$

Ratio b/a	$\{\widetilde{X_n}\}$	$\{\widetilde{r_n}\}$
$15/2$	$A098247$	$A098246$
8	$A097736$	$A097735$
$17/2$	$A098250$	$A098249$
9	$A097739$	$A097738$
$19/2$	$A098253$	$A098252$
10	$A097742$	$A097741$
$21/2$	$A098256$	$A098255$
11	$A097767$	$A097766$
$23/2$	$A098259$	$A098258$
12	$A097770$	$A097769$
$25/2$	$A098262$	$A098261$
13	$A097773$	$A097772$
$27/2$	$A098292$	$A098291$
14	$A097776$	$A097775$

5. Examples

We show now some examples of integer sequences that can be obtained for different values of the parameter b/a.

Example 1. If $b/a = 1/2$, one gets the two following sequences:
$\{\widetilde{X_n}\} = \{1,\ 2,\ 5,\ 13,\ 34,\ 89,\ \dots\}$ that is classified in OEIS as $A001519$;
$\{\widetilde{r_n}\} = \{1,\ 4,\ 11,\ 29,\ 76,\ 199,\ \dots\}$ that is classified in OEIS as $A002878$.

It is interesting to note that $\{\widetilde{X_n}\}$ is composed by a bisection of Fibonacci numbers i.e. F_{2n-1} while $\{\widetilde{r_n}\}$ is composed by a bisection of Lucas numbers $\{L_{2n}\}$.

Example 2. If $b/a = 1$ one gets the two following sequences:
$\{\widetilde{X_n}\} = \{1,\ 5,\ 29,\ 169,\ 985,\ 5741,\ \dots\}$ that is classified in OEIS as $A001653$;
$\{\widetilde{r_n}\} = \{1,\ 7,\ 41,\ 239,\ 1393,\ 8119,\ \dots\}$ that is classified in OEIS as $A002315$.

Reference.

[1] N. J. A. Sloane (editor), *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`.

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