Forum Geometricorum
Volume 19 (2019) 17–28.

ISSN 1534-1178

Adjugate Points and Adjugate Triangle

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Abstract. In this paper we introduce two new notions in triangle geometry: the adjugate points and the adjugate triangle of a point with respect to a given triangle. These notions are used to better characterize the anticomlementary triangles.

1. Preliminaries

Consider a triangle ABC with lengths of sides $BC = a$, $CA = b$, and $AB = c$. Denote by s its semiperimeter and Δ its area. It is well known that the circumradius, inradius, and exradii are given by

$$R = \frac{abc}{4\Delta}, \quad r = \frac{\Delta}{s}, \quad r_a = \frac{\Delta}{s-a}, \quad r_b = \frac{\Delta}{s-b}, \quad r_c = \frac{\Delta}{s-c}.$$

We work with barycentric coordinates, absolute and homogeneous, of points with reference to the triangle. Every *finite* point P is given by its absolute barycentric coordinates $(x_P,\ y_P,\ z_P)$ with $x_P + y_P + z_P = 1$. It is more convenient to work with homogeneous barycentric coordinates. Thus, the same point P is also given by $P = (x :\ y :\ z)$, where $x : y : z = x_P :\ y_P :\ z_P$.

To express the coordinates more succinctly, we also make use of the following notations:

$$S_A = \frac{b^2+c^2-a^2}{2}, \quad S_B = \frac{c^2+a^2-b^2}{2}, \quad S_C = \frac{a^2+b^2-c^2}{2}.$$

These satisfy the following relations, where $S = 2\Delta$, and for convenience, we write S_{BC} for S_BS_C etc.

Publication Date: January 16, 2019. Communicating Editor: Paul Yiu.
The author thanks Editor Paul Yiu for his help in the preparation of this paper.

$$S_A = S\cot A,\ S_B = S\cot B,\ S_C = S\cot C;$$
$$S_{BC} + S_{CA} + S_{AB} = S^2,$$
$$a^2S_A + b^2S_B + c^2S_C = 2S^2,$$
$$a^2S_A + S_{BC} = b^2S_B + S_{CA} = c^2S_C + S_{AB} = S^2;$$
$$bc + ca + ab = s^2 + r^2 + 4Rr,$$
$$a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr),$$
$$a^3 + b^3 + c^3 = 2s(s^2 - 3r^2 - 6Rr).$$

2. Adjugate points and adjugate triangles

Suppose M is a center of triangle ABC, and let $f(a,b,c)$ be a center function for M (given for example by the barycentric coordinates of M):

$$M = (f(a,b,c) :\ f(b,c,a) :\ f(c,a,b)).$$

Definition. The points

$$\begin{aligned} M_a =&\ (f(-a,b,c) : f(b,c,-a) : f(c,-a,b)),\\ M_b =&\ (f(a,-b,c) : f(-b,c,a) : f(c,a,-b)),\\ M_c =&\ (f(a,b,-c) : f(b,-c,a) : f(-c,a,b)) \end{aligned}$$

are called the *adjugate points*, and $\Delta M_aM_bM_c$ the *adjugate triangle* of M with respect to the reference triangle ABC.

Example 1. The adjugate points of the incenter $I = (a : b : c)$ are the centers of excircles, i.e. the points

$$I_a = (-a : b : c),\quad I_b = (a : -b : c),\quad I_c = (a : b : -c).$$

The adjugate triangle of the incenter is the excentral triangle $I_aI_bI_c$ of ABC.

Let X, Y, Z be the tangency points of incircle with the sides of triangle ABC:

$$X = (0 : s-c : s-b),\quad Y = (s-c : 0 : s-a),\quad Z = (s-b : s-a : 0).$$

Similarly we denote the points of tangency of the A-excircle, B-excircle, C-excircle respectively, with the sides of triangle ABC:

$$\begin{array}{lll} X_a = (0 : s-b : s-c), & Y_a = (-(s-b) : 0 : s), & Z_a = (-(s-c) : s : 0);\\ X_b = (0 : -(s-a) : s), & Y_b = (s-a : 0 : s-c), & Z_b = (s : -(s-c) : 0);\\ X_c = (0 : s : -(s-a)), & Y_c = (s : 0 : -(s-b)), & Z_c = (s-a : s-b : 0). \end{array}$$

Theorem 1. *Each of the triplets of lines (AX, BY, CZ), $(AX_a,\ BY_a,\ CZ_a)$, $(AX_b,\ BY_b,\ CZ_b)$, $(AX_c,\ BY_c,\ CZ_c)$ are concurrent. Equivalently, ABC is perspective with each of the triangles XYZ, $X_aY_aZ_a$, $X_bY_bZ_b$, and $X_cY_cZ_c$.* See Figure 1.

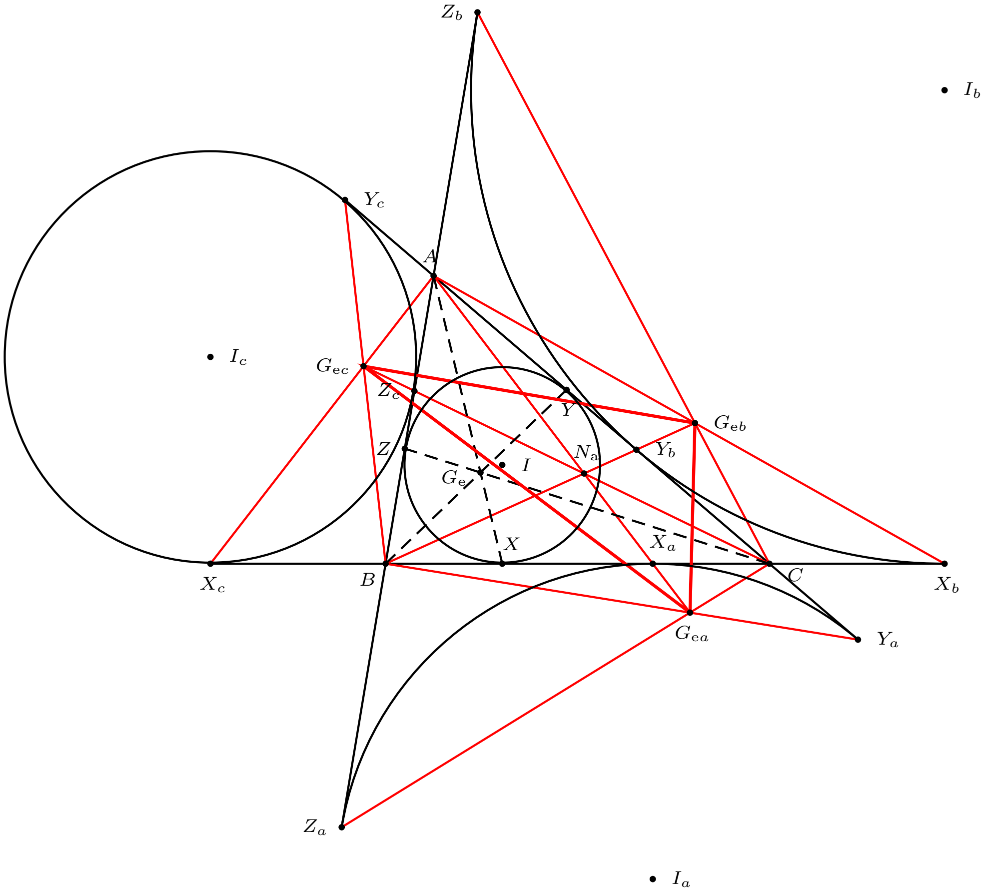

Figure 1. The Gergonne point and its adjugate triangle

Proof. We begin with the equations of the lines:

$AX:\ -(s-b)y+(s-c)z=0,$ $BY:\ (s-a)x-(s-c)z=0,$ $CZ:\ -(s-a)x+(s-b)y=0;$	$AX_a:\ (s-c)y-(s-b)z=0,$ $BY_a:\ (s-b)x+sz=0,$ $CZ_a:\ sx+(s-c)y=0;$
$AX_b:\ sy+(s-a)z=0,$ $BY_b:\ (s-c)x-(s-a)z=0,$ $CZ_b:\ (s-c)x+sy=0;$	$AX_c:\ (s-a)y+sz=0,$ $BY_c:\ (s-c)z+sz=0,$ $CZ_c:\ -(s-b)x+(s-a)y=0.$

It is well known that AX, BY, CZ are concurrent at the Gergonne point

$$G_{\mathrm{e}}=\left(\frac{1}{s-a}:\frac{1}{s-b}:\frac{1}{s-c}\right).$$

With the above equations, it is easy to verify that the concurrency of each of the triples:

Triple of lines		Point of concurrency
$AX_a \cap BY_a \cap CZ_a$		$G_{\mathrm{ea}} = \left(-\frac{1}{s} : \frac{1}{s-c} : \frac{1}{s-b}\right)$
$AX_b \cap BY_b \cap CZ_b$		$G_{\mathrm{eb}} = \left(\frac{1}{s-c} : -\frac{1}{s} : \frac{1}{s-a}\right)$
$AX_c \cap BY_c \cap CZ_c$		$G_{\mathrm{ec}} = \left(\frac{1}{s-b} : \frac{1}{s-a} : -\frac{1}{s}\right)$

This completes the proof of the theorem. The points G_{ea}, G_{eb}, G_{ec} are the adjugate points of the Gergonne point Ge. □

Theorem 2. *Each of the triplets of lines* (AX_a, BY_b, CZ_c), $(AX,\ BY_c,\ CZ_b)$, $(AX_c,\ BY,\ CZ_a)$, $(AX_b,\ BY_a,\ CZ)$ *are concurrent. Equivalently,* ABC *is perspective with each of the triangles* $X_aY_bZ_c$, XY_cZ_b, X_cYZ_a, *and* X_bY_aZ. See Figure 2.

Proof. Again, we begin with the equations of the lines:

$AX_a:\ (s-c)y-(s-b)z=0,$ $BY_b:\ (s-c)x-(s-a)z=0,$ $CZ_c:\ -(s-b)x+(s-a)y=0;$		$AX:\ -(s-b)y+(s-c)z=0,$ $BY_c:\ (s-b)x+sz=0,$ $CZ_b:\ (s-c)x+sy=0;$
$AX_c:\ (s-a)y+sz=0,$ $BY:\ (s-a)x-(s-c)z=0,$ $CZ_a:\ sx+(s-c)y=0;$		$AX_b:\ sy+(s-a)z=0,$ $BY_a:\ sx+(s-b)z=0,$ $CZ:\ -(s-a)x+(s-b)y=0.$

It is well known that AX_a, BY_b, CZ_c are concurrent at the Nagel point

$$N_{\mathrm{a}} = (s-a:\ s-b:\ s-c).$$

The other three points of concurrency are the adjugate points of the Nagel point N_{a}:

Triple of lines		Point of concurrency
$AX \cap BY_c \cap CZ_b$		$N_{\mathrm{aa}} = (-s:\ s-c:\ s-b)$
$AX_c \cap BY \cap CZ_a$		$N_{\mathrm{ab}} = (s-c:\ -s:\ s-a)$
$AX_b \cap BY_a \cap CZ$		$N_{\mathrm{ac}} = (s-b:\ s-a:\ -s)$

This completes the proof of the theorem. The points N_{aa}, N_{ab}, N_{ac} are the adjugate points of the Nagel point Na. □

Theorem 3. *The triangle* ABC *and the adjugate triangle* $G_{\mathrm{ea}}G_{\mathrm{eb}}G_{\mathrm{ec}}$ *of the Gergonne point are perspective, and the perspector is the Nagel point.* See Figure 1.

Proof. The line $AG_{\mathrm{ea}}, BG_{\mathrm{eb}}, CG_{\mathrm{ec}}$ have equations $\frac{y}{s-b} - \frac{z}{s-c} = 0$, $\frac{z}{s-c} - \frac{x}{s-a} = 0$, and $\frac{x}{s-a} - \frac{y}{s-b} = 0$. They are concurrent at $(s-a:\ s-b:\ s-c)$, the Nagel point. □

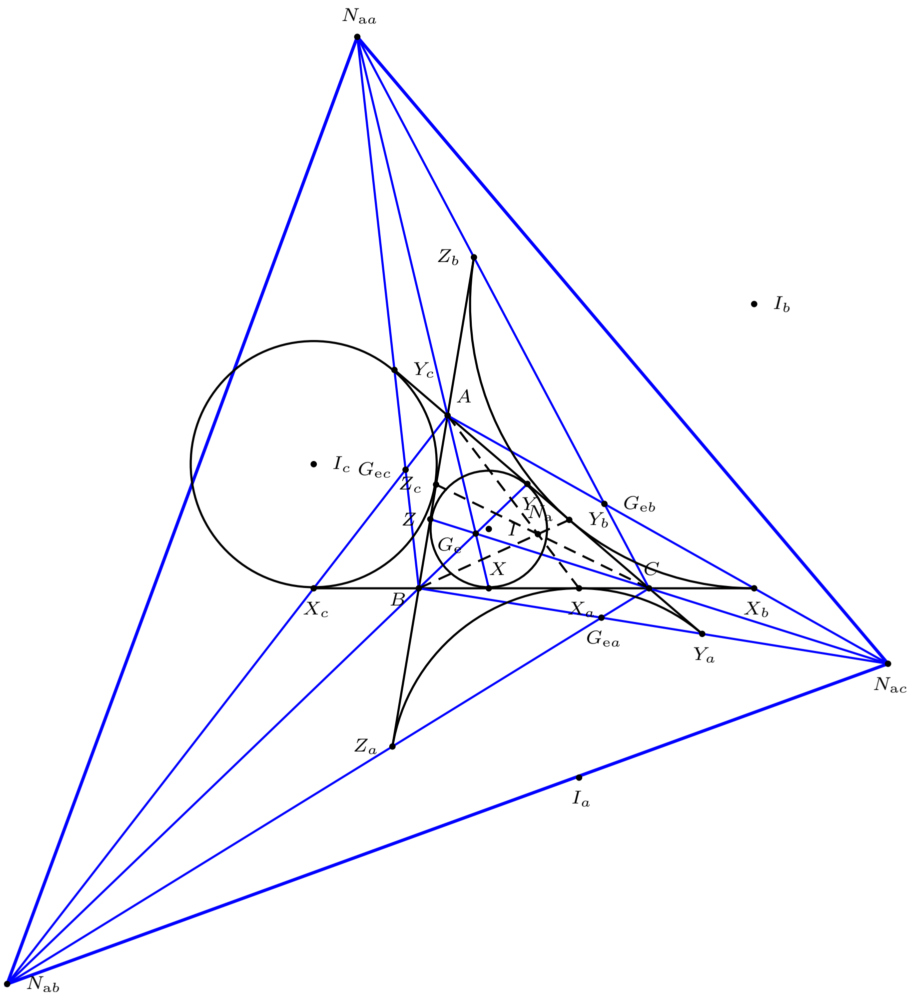

Figure 2. The Nagel point and its adjugate triangle

Theorem 4. *The triangle ABC and the adjugate triangle $N_{\text{aa}}N_{\text{ab}}N_{\text{ac}}$ of the Nagel point are perspective, and the perspector is the Gergonne point.*

Proof. The line AN_{aa}, BN_{ab}, CN_{ac} have equations $(s-b)y-(s-c)z=0$, $(s-c)z-(s-a)x=0$, and $(s-a)x-(s-b)y=0$. They are concurrent at $\left(\frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c}\right)$, the Gergonne point. See Figure 2. □

Example 2. The incircle is internally tangent to the nine-point circle. Its point of tangency is the Feuerbach point F of the triangle ABC:

$$F=((b-c)^2(s-a):\ (c-a)^2(s-b):\ (a-b)^2(s-c)).$$

The nine-point circle is also tangent externally to the excircles at the following points:

$$F_a = \ (-(b-c)^2 s : \ (c+a)^2(s-c) : \ (a+b)^2(s-b)),$$
$$F_b = \ ((b+c)^2(s-c) : \ -(c-a)^2 s : \ (a+b)^2(s-a)),$$
$$F_c = \ ((b+c)^2(s-b) : \ (c+a)^2(s-a) : \ -(a-b)^2 s).$$

Since the coordinates of F can be rewritten as

$$((b-c)^2(b+c-a) : \ (c-a)^2(c+a-b) : \ (a-b)^2(a+b-c)),$$

its A-adjugate point is

$$\begin{aligned} &((b-c)^2(b+c-(-a)) : \ (c-(-a))^2(c+(-a)-b) \\ &\qquad : \ ((-a)-b)^2((-a)+b-c)) \\ =\ &((b-c)^2(a+b+c) : \ -(c+a)^2(a+b-c) : \ -(a+b)^2(c+a-b)) \\ =\ &(-(b-c)^2 s : \ (c+a)^2(s-c) : \ (a+b)^2(s-b)). \end{aligned}$$

This is the point F_a above; similarly for the B- and C-adjugate points. Therefore, the Feuerbach triangle $F_aF_bF_c$ is the adjugate triangle of the Feuerbach point. See Figure 3.

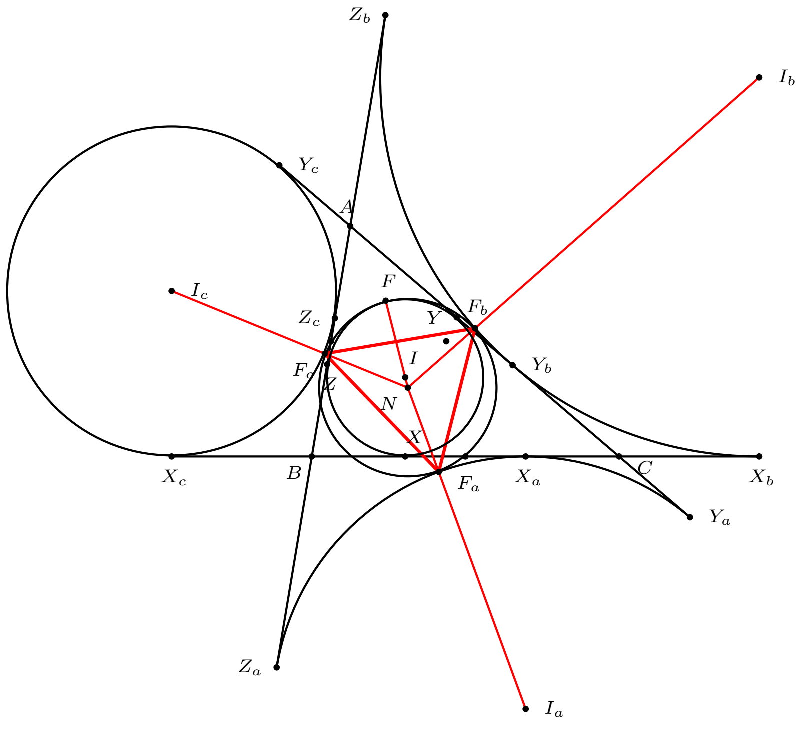

Figure 3. The Feuerbach point and its adjugate triangle

3. Anticomplementary triangle

The anticomplementary triangle is the triangle $A'B'C'$ which has the given triangle ABC as its medial triangle. Consider the homothety h with center in triangle centroid G and ratio -2, i.e. $M'G = 2GM$, $G \in (MM')$. This homothety $\mathsf{h} = \mathsf{h}(G, -2)$ transforms the reference triangle ABC into the anticomplementary triangle $A'B'C'$. Indeed, if (x_M, y_M, z_M) are the absolute barycentric coordinates of M then

$$\mathsf{h}(M) = M' = 3G - 2M = (1 - 2x_M,\ 1 - 2y_M,\ 1 - 2z_M).$$

Therefore,

$$\mathsf{h}(A) = A' = (-1, 1, 1), \quad \mathsf{h}(B) = B' = (1, -1, 1), \quad \mathsf{h}(C) = C' = (1, 1, -1).$$

It is easy to verify that the points A, B, C are the midpoints of segments $B'C'$, $C'A'$, $A'B'$ respectively.

For $\mathsf{h} = \mathsf{h}(G, -2)$, we shall call $M' = \mathsf{h}(M)$ the *anticomplement* of M. In homogeneous barycentric coordinates, if $M = (x : y : z)$, then

$$M' = (-x + y + z : \ x - y + z : \ x + y - z).$$

Here are the coordinates of some common triangle centers and their complements. (The notations follow [2] and [3]).

M	M'
$I = (a : b : c)$	$N_{\mathrm{a}} = (s - a : \ s - b : \ s - c)$
$O = (a^2 S_A : \ b^2 S_B : \ c^2 S_C)$	$H = (S_{BC} : \ S_{CA} : \ S_{AB})$
$H = (S_{BC} : \ S_{CA} : \ S_{AB})$	$X(20) = (S^2 - 2S_{BC} : \ S^2 - 2S_{CA} : \ S^2 - 2S_{AB})$
$N = (S^2 + S_{BC} : \ S^2 + S_{CA} : \ S^2 + S_{AB})$	$O = (a^2 S_A : \ b^2 S_B : \ c^2 S_C)$
$K = (a^2 : \ b^2 : \ c^2)$	$X(69) = (S_A : \ S_B : \ S_C)$
$X(9) = (a(s - a) : \ b(s - b) : \ c(s - c))$	$G_{\mathrm{e}} = \left(\frac{1}{s-a} : \ \frac{1}{s-b} : \ \frac{1}{s-c}\right)$
$S_{\mathrm{p}} = (b + c : \ c + a : \ a + b)$	$I = (a : b : c)$

Theorem 5. *The Euler line of the anticomplementary triangle $A'B'C'$ coincide with the Euler line of the reference triangle ABC .*

Proof. The Euler line is determined by the circumcenter, the orthocenter and the centroid of a triangle. Since $O' = H$, the Euler lines of $A'B'C'$ and ABC coincide. □

Lemma 6 (see [1]). *The anticomplement of the Feuerbach point F is the triangle center*

$$X(100) = \left(\frac{a}{b - c} : \ \frac{b}{c - a} : \ \frac{c}{a - b}\right).$$

Proof. In homogeneous barycentric coordinates,

$$F = ((b - c)^2(s - a) : \ (c - a)^2(s - b) : \ (a - b)^2(s - c)).$$

The A-coordinate of its anticomplement is

$$\begin{aligned}
&-(b-c)^2(s-a)+(c-a)^2(s-b)+(a-b)^2(s-c)\\
=\ &(-(b-c)^2+(c-a)^2+(a-b)^2)s-(-a(b-c)^2+b(c-a)^2+c(a-b)^2)\\
=\ &(2a^2-2ab-2ac+2bc)s-(a^2(b+c)+a(-(b-c)^2-2bc-2bc)+bc^2+b^2c)\\
=\ &2(a-b)(a-c)s-a^2(b+c)+a(b+c)^2-bc(b+c)\\
=\ &2(a-b)(a-c)s-(b+c)(a^2-a(b+c)+bc)\\
=\ &2(a-b)(a-c)s-(b+c)(a-b)(a-c)\\
=\ &(2s-(b+c))(a-b)(a-c)\\
=\ &a(a-b)(a-c).
\end{aligned}$$

Similarly, the B- and C-coordinates are $b(b-c)(b-a)$ and $c(c-a)(c-b)$. Therefore,

$$F'=(a(a-b)(a-c):b(b-c)(b-a):\ c(c-a)(c-b)=\left(\frac{a}{b-c}:\ \frac{b}{c-a}:\ \frac{c}{a-b}\right).$$

□

Lemma 7. *The anticomplement of F_a is the point*

$$F_a'=(-a(a+b)(a+c):\ b(b-c)(b+a):\ c(c-b)(c+a)).$$

Proof. From

$$F_a=\ (-(b-c)^2s:\ (c+a)^2(s-c):\ (a+b)^2(s-b)),$$

the A-coordinate of F_a' is

$$\begin{aligned}
&(b-c)^2s+(c+a)^2(s-c)+(a+b)^2(s-b)\\
=\ &((b-c)^2+(c+a)^2+(a+b)^2)s-(c(c+a)^2+b(a+b)^2)\\
=\ &2(a^2+b^2+c^2-bc+ca+ab)s-(a^2(b+c)+2a(b^2+c^2)+b^3+c^3)\\
=\ &(a+b+c)(a^2+a(b+c)+(b^2-bc+c^2))-(a^2(b+c)+2a(b^2+c^2)+(b^3+c^3))\\
=\ &a^3+a^2(b+c)+a(b^2-bc+c^2)+a^2(b+c)+a(b+c)^2+(b^3+c^3)\\
&-a^2(b+c)-2a(b^2+c^2)-(b^3+c^3)\\
=\ &a^3+a^2(b+c)+abc\\
=\ &a(a+b)(a+c).
\end{aligned}$$

The B-coordinate of the anticomplement of F_a is

$$\begin{aligned}
& -(b-c)^2 s-(c+a)^2(s-c)+(a+b)^2(s-b) \\
= & -(b-c)^2 s-(c+a)^2(s-c)-(a+b)^2(s-b)+2(a+b)^2(s-b) \\
= & -a(a+b)(a+c)+2(a+b)^2(s-b) \\
= & (a+b)[-a(a+c)+2(s-b)(a+b)] \\
= & (a+b)[-a(a+c)+(c+a-b)(a+b)] \\
= & (a+b)[-a^2-ac+a^2-b^2+ac+bc] \\
= & -b(a+b)(b-c).
\end{aligned}$$

Similarly, the C-coordinate is $-c(a+c)(c-b)$. Therefore,

$$F_a' = (-a(a+b)(a+c) : \ b(b-c)(b+a) : \ c(c-b)(c+a)).$$

□

Similarly the anticomplements of F_b and F_c are

$$\begin{aligned}
F_b' = & \ (a(a-c)(a+b) : \ -b(b+c)(b+a) : \ c(c-a)(c+b)), \\
F_c' = & \ (a(a-b)(a+c) : \ b(b-a)(b+c) : \ -c(c+a)(c+b)).
\end{aligned}$$

Lemma 8. *The points F', F_a', F_b', F_c' are on the circumcircle $O(R)$.*

Proof. These points are the anticomplements of F, F_a, F_b, F_c, which are on the nine-point circle. Since the anticomplement of the nine-point circle is the circumcle, the result follows. □

Theorem 9. (a) *The anticomplements of the excenters I_a, I_b, I_c are the adjugate points N_{aa}, N_{ab}, N_{ac} of the Nagel point.*

(b) *The triplets of points (F, N_{a}, O), (F_a', N_{aa}, O), (F_b', N_{ab}, O), (F_c', N_{ac}, O) are collinear, and $FN \parallel OF'$, $F_aN \parallel OF_a'$, $F_bN \parallel OF_b'$, $F_cN \parallel OF_c'$.*

(c) *The circle $N_{\text{a}}(2r)$ is the incircle; the circles $N_{\text{aa}}(2r_a)$, $N_{\text{ab}}(2r_b)$, $N_{\text{ac}}(2r_c)$ are the excircles of the anticomplementary triangle $A'B'C'$.*

(d) *The incircle $N_{\text{a}}(2r)$, and respectively the excircles $N_{\text{aa}}(2r_a)$, $N_{\text{ab}}(2r_b)$, $N_{\text{ac}}(2r_c)$ and the nine-point circle $O(R)$ of the anticomplementary triangle $A'B'C'$ are tangent at the anticomplements of Feuerbach point $F' \equiv X(100)$ and F_a', F_b', F_c'.*

Proof. (a) Since $I_a = (-a : \ b : c)$, we have:

$$\begin{aligned}
I_a' = & \ (-(-a)+b+c : \ -a-b+c : \ -a+b-c) \\
= & \ (a+b+c : \ -(a+b-c) : \ -(c+a-b)) \\
= & \ (-s : \ s-c : \ s-b) \\
= & \ N_{aa}.
\end{aligned}$$

Similarly, $I_b' = N_{\text{a}b}$ and $I_c' = N_{\text{ac}}$.

(b) The Feuerbach point F, the incenter I, and the nine-point center N, are collinear. Since the homothety preserve the collinearity, the points F', N_{a}, and

O as anticomplements of F, I, N are collinear, too. The points F'_a, N_{aa}, O are collinear since they are the images under h of the collinear points F_a, I_a, N. Since $\frac{OG}{GN} = 2 = \frac{F'G}{GF}$, the lines FN and OF' are parallel.

(c) Denote by $D(P, L)$ the distance from the point P to the straight line L. The equations of the sidelines of $A'B'C'$ are:

$$B'C' :\ y + z = 0, \quad C'A' :\ z + x = 0, \quad A'B' :\ x + y = 0.$$

We prove that

$$D(N_{\mathrm{a}},\ B'C') = D(N_{\mathrm{a}},\ C'A') \ = \ D(N_{\mathrm{a}},\ A'B') = 2r$$

and

$$D(N_{\mathrm{aa}},\ B'C') \ = \ D(N_{\mathrm{aa}},\ C'A') \ = \ D(N_{\mathrm{aa}},\ A'B') \ = 2r_a.$$

Indeed,

$$D(N_{\mathrm{a}},\ B'C') = \ S\frac{\left|\frac{s-b}{s} + \frac{s-c}{s}\right|}{\sqrt{b^2 + c^2 - 2S_A}} = \frac{S}{s} = \frac{2sr}{s} = 2r,$$

$$D(N_{\mathrm{aa}},\ B'C') = \ S\frac{\left|-\frac{s-c}{s-a} - \frac{s-b}{s-a}\right|}{\sqrt{b^2 + c^2 - 2S_A}} = \frac{S}{s-a} = \frac{2(s-a)r_a}{s-a} = 2r_a.$$

(d)

$$\begin{aligned}
&(ON_{\mathrm{a}})^2 \\
= \ & R^2 - \frac{1}{s^2}\left[a^2(s-b)(s-c) + b^2(s-c)(s-a) + c^2(s-a)(s-b)\right] \\
= \ & R^2 - \frac{1}{s^2}\left[-s^2(a^2+b^2+c^2) + s(a^3+b^3+c^3) + abc \cdot 2s\right] \\
= \ & R^2 - \frac{1}{s^2}\left[-2s^2(s^2 - r^2 - 4Rr) + 2s^2(s^2 - 3r^2 - 6Rr) + 4Rsr \cdot 2s\right] \\
= \ & R^2 - 2(-2r^2 + 2Rr) \\
= \ & (R-2r)^2;
\end{aligned}$$

i.e. $ON_{\mathrm{a}} = R - 2r$.

Similarly, $ON_{\mathrm{aa}} = R + 2r_a$, $ON_{\mathrm{ab}} = R + 2r_b$, and $ON_{\mathrm{ac}} = R + 2r_c$. □

Remarks. (1) The excentral triangle of the anticomplementary triangle $A'B'C'$ is the adjugate triangle $N_{aa}N_{ab}N_{ac}$ of the Nagel point.

(2) Since homotheties preserve parallelism, the sides of the excentral triangles $I_aI_bI_c$ and $N_{aa}N_{ab}N_{ac}$ are parallel, i.e.

$$I_bI_c \parallel N_{\mathrm{ab}}N_{\mathrm{ac}}, \quad I_cI_a \parallel N_{\mathrm{ac}}N_{\mathrm{aa}}, \quad I_aI_b \parallel N_{\mathrm{aa}}N_{\mathrm{ab}},$$

and

$$N_{\mathrm{ab}}N_{\mathrm{ac}} = 2I_bI_c, \quad N_{\mathrm{ac}}N_{\mathrm{aa}} = 2I_cI_a, \quad N_{\mathrm{aa}}N_{\mathrm{ab}} = 2I_aI_b.$$

Furthermore, we have

$$AI_a \parallel A'N_{\mathrm{aa}}, \quad BI_b \parallel B'N_{\mathrm{ab}}, \quad CI_c \parallel C'N_{\mathrm{ac}}.$$

Corollary 10. *The anticomplementary triangle is the orthic triangle of the adjugate triangle $N_{\rm aa}N_{\rm ab}N_{\rm ac}$ of the Nagel point.*

Theorem 11. *The Feuerbach triangle of the anticomplementary triangle $A'B'C'$ is the adjugate triangle of $X(100)$, the anticomplement of the Feuerbach point F.*

Proof. A center function of F' is $f(a,b,c) = a(a-b)(a-c)$. The points F'_a, F'_b, F'_c are the adjugate points of $X(100)$. Indeed,

$$\begin{aligned} F'_a = &\ (-a(a+b)(a+c) : b(b-c)(b+a) : c(c+a)(c-b)) \\ = &\ (f(-a,b,c) : f(b,c,-a) : f(c,-a,b)). \end{aligned}$$

□

Theorem 12. *The adjugate triangles $G_{\rm ea}G_{\rm eb}G_{\rm ec}$ and $N_{\rm aa}N_{\rm ab}N_{\rm ac}$ of the Gergonne and Nagel points are perspective, and the perspector is the symmedian point of the anticomplementary triangle $A'B'C'$, i.e. the point $K' = X(69)$.*

Proof. It is easy to verify that the lines through the corresponding pairs of points have equations

$$\begin{aligned} a(b-c)sx + b(c+a)(s-c)y - c(a+b)(s-b)z &= 0, \\ -a(b+c)(s-c)x + b(c-a)sy + c(a+b)(s-a)z &= 0, \\ a(b+c)(s-b)x - b(c+a)(s-a)y + c(a-b)sz &= 0, \end{aligned}$$

and that each of these lines contains the point

$$X(69) = (b^2+c^2-a^2 :\ c^2+a^2-b^2 :\ a^2+b^2-c^2).$$

□

Theorem 13. *The Gergonne point $G_{\rm e}$, the symmedian point of the anticomplementary triangle K', and the Nagel point $N_{\rm a}$ are collinear.*

Proof. The line containing these points has equation

$$a(b-c)(s-a)x + b(c-a)(s-b)y + c(a-b)(s-c)z = 0.$$

□

Theorem 14. *The pairs of perspective triangles*

$$(ABC,\ G_{\rm ea}G_{\rm eb}G_{\rm ec}) \quad (ABC,\ N_{\rm aa}N_{\rm ab}N_{\rm ac}) \quad (G_{\rm ea}G_{\rm eb}G_{\rm ec},\ N_{\rm aa}N_{\rm ab}N_{\rm ac})$$

have a common perspectrix, which is the trilinear polar of the isotomic conjugate of incenter with equation $ax + by + cz = 0$.

Proof. (a) The sidelines of triangle $G_{\rm ea}G_{\rm eb}G_{\rm ec}$ have equations

$$\begin{aligned} G_{\rm eb}G_{\rm ec}: &\quad -(b+c)(s-b)(s-c)x + bs(s-a)y + cs(s-a)z = 0, \\ G_{\rm ec}G_{\rm ea}: &\quad as(s-b)x - (c+a)(s-c)(s-a)y + cs(s-b)z = 0, \\ G_{\rm ea}G_{\rm eb}: &\quad as(s-c)x + bs(s-c)y - (a+b)(s-a)(s-b)z = 0. \end{aligned}$$

These lines intersect BC, CA, AB respectively at the three points

$$(0 :\ c :\ -b), \quad (-c :\ 0 :\ a), \quad (b :\ -a :\ 0)$$

collinear on the line $ax + by + cz = 0$. This shows that ABC and $G_{\mathrm{ea}}G_{\mathrm{eb}}G_{\mathrm{ec}}$ are perspective with perspectrix the trilinear polar of $\left(\frac{1}{a} : \frac{1}{b} : \frac{1}{c}\right)$.

(b) The sidelines of triangle $N_{\mathrm{aa}}N_{\mathrm{ab}}N_{\mathrm{ac}}$ have equations

$$\begin{aligned} N_{\mathrm{ab}}N_{\mathrm{ac}}: &\qquad (b+c)x + by + cz = 0, \\ N_{\mathrm{ac}}N_{\mathrm{aa}}: &\qquad ax + (c+a)y + cz = 0, \\ N_{\mathrm{aa}}N_{\mathrm{ab}}: &\qquad ax + by + (a+b)z = 0. \end{aligned}$$

These lines intersect BC, CA, AB respectively at the same three points

$$(0 : c : -b), \quad (-c : 0 : a), \quad (b : -a : 0)$$

on the line $ax + by + cz = 0$, showing that ABC and $N_{\mathrm{aa}}N_{\mathrm{ab}}N_{\mathrm{ac}}$ are perspective with the same perspectrix.

(c) For the triangles $G_{\mathrm{ea}}G_{\mathrm{eb}}G_{\mathrm{ec}}$ and $N_{\mathrm{aa}}N_{\mathrm{ab}}N_{\mathrm{ac}}$, we have

$$\begin{aligned} G_{\mathrm{eb}}G_{\mathrm{ec}} \cap N_{\mathrm{ab}}N_{\mathrm{ac}} &= (0 : c : -b), \\ G_{\mathrm{ec}}G_{\mathrm{ea}} \cap N_{\mathrm{ac}}N_{\mathrm{aa}} &= (-c : 0 : a), \\ G_{\mathrm{ea}}G_{\mathrm{eb}} \cap N_{\mathrm{aa}}N_{\mathrm{ab}} &= (b : -a : 0). \end{aligned}$$

Again, the corresponding lines intersect at the same three collinear points on $ax + by + cz = 0$.

This shows that the three pairs of perspective triangles have the same perspectrix. □

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