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Heptagonal Triangle and Trigonometric Identities

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Abstract. We will study the trigonometric identities for heptagonal triangles. Let $a < b < c$ be the heptagonal triangle's sides and let R be the circumradius. We will prove the following:

$$2b^2 - a^2 = \sqrt{7}bR, \quad 2c^2 - b^2 = \sqrt{7}cR, \quad 2a^2 - c^2 = -\sqrt{7}aR.$$

We will also prove the following trigonometric formula:

$$4\sin\frac{2k\pi}{7} - \tan\frac{k\pi}{7} = \begin{cases} \sqrt{7} & \text{for } k = 1, 2, 4, \\ -\sqrt{7} & \text{for } k = 3, 5, 6., \end{cases}$$

1. Introduction

In this paper, for convenience, let $\theta = \pi/7$. A heptagonal triangle is an obtuse scalene triangle whose vertexes coincide with the first, second, and fourth vertexes of a regular heptagon. Its angles have measures $\theta, 2\theta, 4\theta$. Let $a < b < c$ be the heptagonal triangle's sides and let R be the circumradius. We will prove the following:

Theorem 1. *With above notations, we have*

$$2b^2 - a^2 = \sqrt{7}bR, \quad 2c^2 - b^2 = \sqrt{7}cR, \quad 2a^2 - c^2 = -\sqrt{7}aR.$$

This result is a corollary of the following identities:

Theorem 2.

$$4\sin\frac{2k\pi}{7} - \tan\frac{k\pi}{7} = \begin{cases} \sqrt{7} & \textit{for } k = 1, 2, 4, \\ -\sqrt{7} & \textit{for } k = 3, 5, 6. \end{cases}$$

The purpose of this paper is to prove our results. In later sections, we will also show how to use our methods to prove some known identities which are sums of mixed powers of sine values.

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2. Sums of sine powers

We start with the following theorem which can be proved easily from trigonometric identities from [1, 5].

Theorem 3. (1) $\{\sin 2\theta, \sin 4\theta, \sin 8\theta\}$ *are the roots of*

$$x^3 - \frac{\sqrt{7}}{2}x^2 + \frac{\sqrt{7}}{8} = 0.$$

(2) $\tan\theta \tan 2\theta \tan 4\theta = \tan\theta + \tan 2\theta + \tan 4\theta = -\sqrt{7}.$

(3) $\sec 2\theta + \sec 4\theta + \sec 8\theta = -4.$

Definition. For an integer n, let

$$S(n) = \sin^n 2\theta + \sin^n 4\theta + \sin^n 8\theta.$$

Proposition 4. $S(n)$ *satisfies the recurrence relation:*

$$S(n) = \frac{\sqrt{7}}{2}S(n-1) - \frac{\sqrt{7}}{8}S(n-3).$$

? *Proof.* This follows easily from Theorem 1 □

Now using recurrence relation we can compute $S(n)$ for any integer n. In the following, we will only show a few terms which will be used in later applications.

Example 1. With above notations, the values of $S(n)$ for $n = 1, \ldots,$ 20 are as follows.

n	0	1	2	3	4	5	6
$S(n)$	3	$\frac{\sqrt{7}}{2}$	$\frac{7}{2^2}$	$\frac{\sqrt{7}}{2}$	$\frac{7\cdot 3}{2^4}$	$\frac{7\sqrt{7}}{2^4}$	$\frac{7\cdot 5}{2^5}$
$S(-n)$	3	0	2^3	$-\frac{2^3\cdot 3\sqrt{7}}{7}$	2^5	$-\frac{2^5\cdot 5\sqrt{7}}{7}$	$\frac{2^6\cdot 17}{7}$
n	7	8	9	10	11	12	13
$S(n)$	$\frac{7^2\sqrt{7}}{2^7}$	$\frac{7^2\cdot 5}{2^8}$	$\frac{7\cdot 25\sqrt{7}}{2^9}$	$\frac{7^2\cdot 9}{2^9}$	$\frac{7^2\cdot 13\sqrt{7}}{2^{11}}$	$\frac{7^2\cdot 33}{2^{11}}$	$\frac{7^2\cdot 3\sqrt{7}}{2^9}$
$S(-n)$	$-2^7\sqrt{7}$	$\frac{2^9\cdot 11}{7}$	$-\frac{2^{10}\cdot 33\sqrt{7}}{7^2}$	$\frac{2^{10}\cdot 29}{7}$	$-\frac{2^{14}\cdot 11\sqrt{7}}{7^2}$	$\frac{2^{12}\cdot 269}{7^2}$	$-\frac{2^{13}\cdot 117\sqrt{7}}{7^2}$
n	14	15	16	17	18	19	20
$S(n)$	$\frac{7^4\cdot 5}{2^{14}}$	$\frac{7^2\cdot 179\sqrt{7}}{2^{15}}$	$\frac{7^3\cdot 131}{2^{16}}$	$\frac{7^3\cdot 3\sqrt{7}}{2^{12}}$	$\frac{7^3\cdot 493}{2^{18}}$	$\frac{7^3\cdot 181\sqrt{7}}{2^{18}}$	$\frac{7^5\cdot 19}{2^{19}}$
$S(-n)$	$\frac{2^{14}\cdot 51}{7}$	$-\frac{2^{21}\cdot 17\sqrt{7}}{7^3}$	$\frac{2^{17}\cdot 237}{7^2}$	$-\frac{2^{17}\cdot 1445\sqrt{7}}{7^3}$	$\frac{2^{19}\cdot 2203}{7^3}$	$-\frac{2^{19}\cdot 1919\sqrt{7}}{7^3}$	$\frac{2^{20}\cdot 5851}{7^3}$

3. Lemmas

Definition. For integers, m, n, let

$$W(m,n) = \sin^m 2\theta \sin^n 4\theta + \sin^m 4\theta \sin^n 8\theta + \sin^m 8\theta \sin^n 2\theta$$

and let

$$P = \sin 2\theta \sin 4\theta \sin 8\theta = -\frac{\sqrt{7}}{8}.$$

Lemma 5.

$$\begin{aligned} W(m.n)+W(n,m) =&\ S(m)S(n)-S(m+n),\\ W(m.n)W(n,m) =&\ P^{m+n}S(-(m+n))+P^mS(2n-m)+P^nS(2m-n). \end{aligned}$$

Proof. This can be proved easily using simple algebra. □

Remark. Note that if $m \neq n$, $W(m,n)$ is not a symmetrical polynomial in $\{\sin 2\theta, \sin 4\theta, \sin 8\theta\}$ and in general, it is not easy to compute $W(m,n)$ directly. Here is our approach. Using Lemma 5, we can compute $W(m.n)+W(n,m)$ and $W(m.n)W(n,m)$ in terms of $S(n)$ and P. Then we solve a quadratic equation

$$x^2-(W(m.n)+W(n,m))x+W(m.n)W(n,m)=0.$$

and use approximate values to identify the solutions.

Lemma 6.

$$W(2,3)=\frac{7\sqrt{7}}{32}.$$

Proof. By Lemma 5,

$$\begin{aligned} W(2,3)+W(3,2) =&\ S(2)S(3)-S(5)=\frac{7\sqrt{7}}{16},\\ W(2,3)W(3,2) =&\ P^5S(-5)+P^2S(4)+P^3S(1)=\frac{343}{1024}. \end{aligned}$$

Solving the quadratic equation

$$t^2-\frac{7\sqrt{7}}{16}t+\frac{343}{1024}=0,$$

we have

$$t=\left\{\frac{7\sqrt{7}}{32},\frac{7\sqrt{7}}{32}\right\}.$$

This proves this lemma. □

Definition. For convenience let

$$\begin{aligned} R =&\ \sin 2\theta\sin 4\theta\tan 8\theta+\sin 4\theta\sin 8\theta\tan 2\theta+\sin 8\theta\sin 2\theta\tan 4\theta,\\ U =&\ \sin 2\theta\sin 4\theta\tan 4\theta+\sin 4\theta\sin 8\theta\tan 8\theta+\sin 8\theta\sin 2\theta\tan 2\theta,\\ V =&\ \sin 2\theta\sin 8\theta\tan 8\theta+\sin 4\theta\sin 2\theta\tan 2\theta+\sin 8\theta\sin 4\theta\tan 4\theta,\\ X =&\ \sin^2 2\theta\tan 2\theta+\sin^2 4\theta\tan 4\theta+\sin^2 8\theta\tan 8\theta,\\ Y =&\ \sin^2 2\theta\tan 8\theta+\sin^2 4\theta\tan 2\theta+\sin^2 8\theta\tan 4\theta,\\ Z =&\ \sin^2 2\theta\tan 4\theta+\sin^2 4\theta\tan 8\theta+\sin^2 8\theta\tan 2\theta. \end{aligned}$$

Lemma 7.

$$R=\frac{\sqrt{7}}{2},\quad V=\sqrt{7},\quad U=-\frac{3\sqrt{7}}{2},\quad X=-\frac{5\sqrt{7}}{4}.$$

Proof. With above notations,

$$\begin{aligned}
R = & \sin 2\theta \sin 4\theta \tan 8\theta + \sin 4\theta \sin 8\theta \tan 2\theta + \sin 8\theta \sin 2\theta \tan 4\theta \\
= & (\sin 2\theta \sin 4\theta \sin 8\theta)(\frac{1}{\cos 2\theta} + \frac{1}{\cos 4\theta} + \frac{1}{\cos 8\theta}) \\
= & (\sin 2\theta \sin 4\theta \sin 8\theta)(\sec 2\theta + \sec 4\theta + \sec 8\theta) \\
= & \frac{\sqrt{7}}{2} \\
V = & \sin 2\theta \sin 4\theta \tan 2\theta + \sin 4\theta \sin 8\theta \tan 4\theta + \sin 8\theta \sin 2\theta \tan 8\theta \\
= & 2(\sin^3 2\theta + \sin^3 4\theta + \sin^3 8\theta) \\
= & 2S(3) \\
= & \sqrt{7}; \\
R + U + V = & (\sin 2\theta \sin 4\theta + \sin 4\theta \sin 8\theta + \sin 8\theta \sin 2\theta) \\
& \cdot (\tan 2\theta + \tan 4\theta + \tan 8\theta) \\
= & 0; \\
U = & -R - V \\
= & -\frac{3\sqrt{7}}{2}; \\
X = & \sin^2 2\theta \tan 2\theta + \sin^2 4\theta \tan 4\theta + \sin^2 8\theta \tan 8\theta. \\
= & (1 - \cos^2 2\theta) \tan 2\theta + (1 - \cos^2 4\theta) \tan 4\theta + (1 - \cos^2 8\theta) \tan 8\theta \\
= & (\tan 2\theta + \tan 4\theta + \tan 8\theta) - \frac{1}{2} S(1) \\
= & -\frac{5\sqrt{7}}{4}.
\end{aligned}$$

□

Lemma 8.

$$Y = \frac{\sqrt{7}}{4}, \qquad Z = -\frac{3\sqrt{7}}{4}.$$

Proof.

$$\begin{aligned}
Y + Z = & (\sin^2 2\theta + \sin^2 4\theta + \sin^2 8\theta)(\tan 2\theta + \tan 4\theta + \tan 8\theta) - X \\
= & S(2)(\tan 2\theta + \tan 4\theta + \tan 8\theta)) + \frac{5\sqrt{7}}{4} \\
= & -\frac{\sqrt{7}}{2}.
\end{aligned}$$

Next, we compute

$$\begin{aligned}
&4Y - Z \\
= &(4\sin^2 4\theta - \sin^2 8\theta)\tan 2\theta + (4\sin^2 8\theta - \sin^2 2\theta)\tan 4\theta \\
&+ (4\sin^2 2\theta - \sin^8\theta)\tan 8\theta \\
= &(4\sin^2 4\theta - 4\sin^2 4\theta\cos^2 4\theta)\tan 2\theta + (4\sin^2 8\theta - 4\sin^2 8\theta\cos^2 8\theta)\tan 4\theta \\
&+ (4\sin^2 2\theta - 4\sin^2 2\theta\cos^2 2\theta)\tan 8\theta \\
= &4\sin^2 4\theta(1-\cos^2 4\theta)\tan 2\theta + 4\sin^2 8\theta(1-\cos^2 8\theta)\tan 4\theta \\
&+ 4\sin^2 2\theta(1-\cos^2 2\theta)\tan 8\theta \\
= &4(\sin^4 4\theta\tan 2\theta + \sin^4 8\theta\tan 4\theta + \sin^4 2\theta\tan 8\theta) \\
= &4(2\sin 2\theta\cos 2\theta\tan 2\theta\sin^3 4\theta + 2\sin 4\theta\cos 4\theta\tan 4\theta\sin^3 8\theta \\
&+ 2\sin 8\theta\cos 8\theta\tan 8\theta\sin^3 2\theta) \\
= &8(\sin^2 2\theta\sin^3 4\theta + \sin^2 4\theta\sin^3 8\theta + \sin^2 8\theta\sin^3 2\theta) \\
= &8W(2,3) \\
= &\frac{7\sqrt{7}}{4}.
\end{aligned}$$

It follows that

$$Y = \frac{\sqrt{7}}{4}, \qquad Z = -\frac{3\sqrt{7}}{4}.$$

□

4. Main theorem

Proposition 9. *With above notations,*

$$\begin{aligned}
\tan 2\theta &= -2\sin 2\theta + 2\sin 4\theta - 2\sin 8\theta, \\
\tan 4\theta &= -2\sin 2\theta - 2\sin 4\theta + 2\sin 8\theta, \\
\tan 8\theta &= 2\sin 2\theta - 2\sin 4\theta - 2\sin 8\theta.
\end{aligned}$$

Proof. First we consider a system of linear equations:

$$\begin{aligned}
\tan 2\theta &= x\sin 2\theta + y\sin 4\theta + z\sin 8\theta, \\
\tan 4\theta &= x\sin 4\theta + y\sin 8\theta + z\sin 2\theta, \\
\tan 8\theta &= x\sin 8\theta + y\sin 2\theta + z\sin 4\theta.
\end{aligned}$$

Note that by adding up the equations

$$x + y + z = \frac{\tan 2\theta + \tan 4\theta + \tan 8\theta}{\sin 2\theta + \sin 4\theta + \sin 8\theta} = -2.$$

Then

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}, \quad z = \frac{\Delta_z}{\Delta}$$

where

$$\Delta = DET \begin{bmatrix} \sin 2\theta & \sin 4\theta & \sin 8\theta \\ \sin 4\theta & \sin 8\theta & \sin 2\theta \\ \sin 8\theta & \sin 2\theta & \sin 4\theta \end{bmatrix},$$

$$\Delta_x = DET \begin{bmatrix} \tan 2\theta & \sin 4\theta & \sin 8\theta \\ \tan 4\theta & \sin 8\theta & \sin 2\theta \\ \tan 8\theta & \sin 2\theta & \sin 4\theta \end{bmatrix},$$

$$\Delta_y = DET \begin{bmatrix} \sin 2\theta & \tan 2\theta & \sin 8\theta \\ \sin 4\theta & \tan 4\theta & \sin 2\theta \\ \sin 8\theta & \tan 8\theta & \sin 4\theta \end{bmatrix},$$

$$\Delta_z = DET \begin{bmatrix} \sin 2\theta & \sin 4\theta & \tan 2\theta \\ \sin 4\theta & \sin 8\theta & \tan 4\theta \\ \sin 8\theta & \sin 2\theta & \tan 8\theta \end{bmatrix}.$$

Then by expanding the determinants,

$$\begin{aligned} \Delta &= 3P - S(3) = -\frac{7\sqrt{7}}{8}; \\ \Delta_y &= U - Y = \frac{-3\sqrt{7}}{2} - \frac{\sqrt{7}}{4} = -\frac{7\sqrt{7}}{2}, \\ y &= \frac{\Delta_y}{\Delta} = 2; \\ \Delta_z &= V - Z = \sqrt{7} + \frac{3\sqrt{7}}{4} = \frac{7\sqrt{7}}{2}, \\ z &= \frac{\Delta_z}{\Delta} = -2. \end{aligned}$$

Finally,

$$x = -2 - y - z = -2.$$

□

Now we can prove Theorem 2.

Proof. By Proposition 9

$$\begin{aligned} \tan 2\theta &= -2\sin 2\theta + 2\sin 4\theta - 2\sin 8\theta \\ &= 4\sin 4\theta - 2(\sin 2\theta + 2\sin 4\theta + 2\sin 8\theta) \\ &= 4\sin 4\theta - \sqrt{7}. \end{aligned}$$

Similarly, we can prove other identities. □

5. The heptagonal triangle

The heptagonal triangle and trigonometric identities for angles $\theta, 2\theta, 4\theta$ of the heptagonal triangle have been studied in [1, 5]. We will use some results from [1, 5].

Proposition 10. *With above notations, we have*

(1) $\frac{a}{\sin\theta} = \frac{b}{\sin 2\theta} = \frac{c}{\sin 4\theta} = 2R$;
(2) $\cos\theta = \frac{b}{2a}, \quad \cos 2\theta = \frac{c}{2b}, \quad \cos 4\theta = -\frac{a}{2c}$;
(3) $b^2 - a^2 = ca, \quad c^2 - b^2 = ab, \quad a^2 - c^2 = -bc$.

Now we prove Theorem 1 as follows.

Proof. By Theorem 2 and Proposition 10

$$\sin 2\theta = \frac{b}{2R}, \quad \sin\theta = \frac{a}{2R}, \quad \cos\theta = \frac{b}{2a}.$$

It follows that

$$4\sin 2\theta - \tan\theta = \frac{2b}{R} - \frac{a^2}{bR}.$$

$$2b^2 - a^2 = \sqrt{7}bR.$$

Similarly, we can prove other identities. □

6. More trigonometric identities

Proposition 11. *With above notations,*

(1) $\sin^3 2\theta \sin 4\theta + \sin^3 4\theta \sin 8\theta + \sin^3 8\theta \sin 2\theta = 0,$
(2) $\sin 2\theta \sin^3 4\theta + \sin 4\theta \sin^3 8\theta + \sin 8\theta \sin^3 2\theta = \frac{7}{2^4},$
(3) $\sin^4 2\theta \sin 4\theta + \sin^4 4\theta \sin 8\theta + \sin^4 8\theta \sin 2\theta = 0,$
(4) $\sin 2\theta \sin^4 4\theta + \sin 4\theta \sin^4 8\theta + \sin 8\theta \sin^4 2\theta = \frac{7\sqrt{7}}{2^5},$
(5) $\sin^{11} 2\theta \sin^3 4\theta + \sin^{11} 4\theta \sin^3 8\theta + \sin^{11} 8\theta \sin^3 2\theta = 0,$
(6) $\sin^3 2\theta \sin^{11} 4\theta + \sin^3 4\theta \sin^{11} 8\theta + \sin^3 8\theta \sin^{11} 2\theta = \frac{7^3 \cdot 17}{2^{14}}.$

Proof. To prove equations (1) and (2), by Lemma 5,

$$\begin{aligned}
W(3,1) + W(1,3) &= S(3)S(1) - S(4) \\
&= \frac{\sqrt{7}}{2} \cdot \frac{\sqrt{7}}{2} - \frac{7 \cdot 3}{2^4} \\
&= \frac{7}{2^4}; \\
W(3,1)W(1,3) &= P^4 S(-4) + P^3 S(-1) + P^1 S(5) \\
&= \frac{7^2}{2^{12}} \cdot 2^5 - \frac{7\sqrt{7}}{2^9} \cdot 0 - \frac{\sqrt{7}}{2^3} \cdot \frac{7\sqrt{7}}{2^4} \\
&= 0.
\end{aligned}$$

Then solving the quadratic equation

$$x^2 - \frac{7}{16}x = 0,$$

we have roots $\{0, \frac{7}{16}\}$.

Using approximate values, we can conclude that

$$W(3,1) = 0, \quad W(1,3) = \frac{7}{16}.$$

Similarly, to prove equations (3), (4), we compute

$$\begin{aligned} W(4,1) + W(1,4) &= S(4)S(1) - S(5) \\ &= \frac{7 \cdot 3}{2^4} \cdot \frac{\sqrt{7}}{2} - \frac{7\sqrt{7}}{2^4} \\ &= \frac{7\sqrt{7}}{2^5}; \\ W(4,1)W(1,4) &= P^5S(-5) + P^4S(-2) + P^1S(7) \\ &= \frac{-7^2\sqrt{7}}{2^{15}} \cdot \frac{-2^5 \cdot 5\sqrt{7}}{7} + \frac{7^2}{2^{12}} \cdot 2^3 + \frac{-\sqrt{7}}{2^3} \cdot \frac{7^2\sqrt{7}}{2^7} \\ &= \frac{7^2 \cdot 5}{2^{10}} + \frac{7^2}{2^9} + \frac{-7^3}{2^{10}} = 0. \end{aligned}$$

Similarly by solving the quadratic equation

$$x^2 - \frac{7\sqrt{7}}{2^5}x = 0$$

and numerical computations, we have that

$$W(4,1) = 0, \quad W(1,4) = \frac{7\sqrt{7}}{2^5}.$$

Similarly, to prove equations (5) and (6), we compute

$$\begin{aligned} W(11,3) + W(3,11) &= S(11)S(3) - S(14) \\ &= \frac{7^2 \cdot 13\sqrt{7}}{2^{11}} \cdot \frac{\sqrt{7}}{2} - \frac{7^4 \cdot 5}{2^{14}} \\ &= \frac{7^3 \cdot 17}{2^{14}}; \\ W(11,3)W(3,11) &= P^{14}S(-14) + P^{11}S(-5) + P^3S(19) \\ &= \frac{7^7}{2^{42}} \cdot \frac{2^{14} \cdot 51}{7} + \frac{-7^5\sqrt{7}}{2^{33}} \cdot \frac{-2^5 \cdot 5\sqrt{7}}{7} + \frac{-7\sqrt{7}}{2^9} \cdot \frac{7^3 \cdot 181\sqrt{7}}{2^{18}} \\ &= \frac{7^6 \cdot 51}{2^{28}} + \frac{7^5 \cdot 5}{2^{28}} + \frac{-7^5 \cdot 181}{2^{27}} \\ &= 0. \end{aligned}$$

Similarly by solving the quadratic equation

$$x^2 - \frac{7^3 \cdot 17}{2^{14}} x = 0$$

and numerical computations, we have that

$$W(11,3) = 0, \quad W(3,11) = \frac{7^3 \cdot 17}{2^{14}}.$$

□

To further illustrate our method, we will prove the following identities.

Proposition 12.

$$\frac{\sin 2\theta}{\sin^4 4\theta} + \frac{\sin 4\theta}{\sin^4 8\theta} + \frac{\sin 8\theta}{\sin^4 2\theta} = \frac{72\sqrt{7}}{7},$$
$$\frac{\sin 4\theta}{\sin^4 2\theta} + \frac{\sin 8\theta}{\sin^4 4\theta} + \frac{\sin 2\theta}{\sin^4 8\theta} = \frac{64\sqrt{7}}{7}.$$

Proof.

$$\begin{aligned} W(1,-4) + W(-4,1) = &\ S(1)S(-4) - S(-3) = \frac{\sqrt{7}}{2} \cdot 2^5 + \frac{2^3 \cdot 3\sqrt{7}}{7} = \frac{2^3 \cdot 17\sqrt{7}}{7}. \\ W(1,-4)W(-4,1) = &\ P^{-3}S(3) + P^1 S(-9) + P^{-4} S(6) \\ = &\ \frac{-2^9\sqrt{7}}{7^2} \cdot \frac{\sqrt{7}}{2} + \frac{-\sqrt{7}}{2^3} \cdot \frac{-2^{10} \cdot 33\sqrt{7}}{7^2} + \frac{2^{12}}{7^2} \cdot \frac{7 \cdot 5}{2^5} \\ = &\ \frac{-2^8}{7} + \frac{2^7 \cdot 33}{7} + \frac{2^7 \cdot 5}{7} \\ = &\ \frac{2^9 \cdot 9}{7}. \end{aligned}$$

Solving the quadratic equation

$$x^2 + \frac{136\sqrt{7}}{7} x - \frac{4068}{7} = 0$$

we have solutions

$$\{x = \frac{64\sqrt{7}}{7}, \frac{72\sqrt{7}}{7}\}.$$

□

7. The heptagonal triangle again

Theorem 13. *With above notations, we have*

(1) $a^3 b - b^3 c + c^3 a = 0$.
(2) $a^4 b + b^4 c - c^4 a = 0$.
(3) $a^{11} b^3 - b^{11} c^3 + c^{11} a^3 = 0$.
(4) $a^3 c + b^3 a - c^3 b = -7R^4$.
(5) $a^4 c - b^4 a + c^4 b = 7\sqrt{7} R^5$.
(6) $a^{11} c^3 + b^{11} a^3 - c^{11} b^3 = -7^3 \cdot 17 R^{14}$.

Proof. By [1],

$$a = 2R\sin\theta, b = 2R\sin 2\theta, c = 2R\sin 4\theta.$$

Then Theorem 13 follows easily from Proposition 11. □

8. Other related results

Remark. In [2, 3, 4] there are similar trigonometric identities which were proved as corollaries of theta function identities.

$$\frac{\sin 2\theta}{\sin\theta} - \frac{\sin 3\theta}{\sin 2\theta} + \frac{\sin\theta}{\sin 3\theta} = 1,$$
$$\frac{\sin\theta}{\sin 2\theta} - \frac{\sin 2\theta}{\sin 3\theta} + \frac{\sin 3\theta}{\sin\theta} = 2,$$
$$\frac{\sin^2\theta}{\sin 3\theta} - \frac{\sin^2 2\theta}{\sin\theta} + \frac{\sin^2 3\theta}{\sin 2\theta} = 0,$$
$$\frac{\sin 2\theta}{\sin^4\theta} - \frac{\sin\theta}{\sin^4 3\theta} + \frac{\sin 3\theta}{\sin^4 2\theta} = \frac{64\sqrt{7}}{7},$$
$$\frac{\sin^4 3\theta}{\sin\theta} - \frac{\sin^4\theta}{\sin 2\theta} - \frac{\sin^4 2\theta}{\sin 3\theta} = \frac{5\sqrt{7}}{8},$$
$$\frac{\sin^7 2\theta}{\sin^7\theta} - \frac{\sin^7 3\theta}{\sin^7 2\theta} + \frac{\sin^7\theta}{\sin^7 3\theta} = 57,$$
$$\frac{\sin^7\theta}{\sin^7 2\theta} - \frac{\sin^7 2\theta}{\sin^7 3\theta} + \frac{\sin^7 3\theta}{\sin^7\theta} = 289,$$
$$\frac{\sin^3 3\theta}{\sin^6\theta} - \frac{\sin^3\theta}{\sin^6 2\theta} + \frac{\sin^3 2\theta}{\sin^6 3\theta} = \frac{368}{\sqrt{7}},$$
$$\frac{\sin 2\theta}{\sin^2 3\theta} - \frac{\sin\theta}{\sin^2 2\theta} + \frac{\sin 3\theta}{\sin^2\theta} = 2\sqrt{7},$$
$$\csc^7\theta - \csc^7 2\theta - \csc^7 3\theta = 2^7\sqrt{7}.$$

All those identities can be easily proved using our method.

References

[1] L. Bankoff and J. Garfunkel, The heptagonal triangle, *Math. Mag.*, 46 (1973), 7–19.

[2] B. C. Berndt and L. C. Zhang, Ramanujan's identities for eta functions, *Math. Ann.*, 292 (1992), no. 3, 561–573.

[3] B. C. Berndt and A. Zaharescu, Finite trigonometric sums and class numbers, *Math. Ann.*, 330 (2004), no. 3, 551–575.

[4] Z. G. Liu, Some Eisenstein series identities related to modular equations of the seventh order, *Pacific J. Math.*, 209 (2003) no. 1, 103–130.

[5] P. Yiu, Heptagonal triangles and their companions, *Forum Geom.*, 9 (2009) 125–148.

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