Forum Geometricorum
Volume 19 (2019) 39–43.

FORUM GEOM
ISSN 1534-1178

The Radical Axis of the Circumcircle and Incircle of a Bicentric Quadrilateral

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Abstract. We present an unusual method to identify the radical axis of the circumcircle and incircle of a bicentric quadrilateral. Along the way, we demonstrate a number of interesting properties of the configuration.

1. Bicentric Quadrilateral

Let $ABCD$ be a bicentric quadrilteral (see Figure 1). We employ the following definitions:

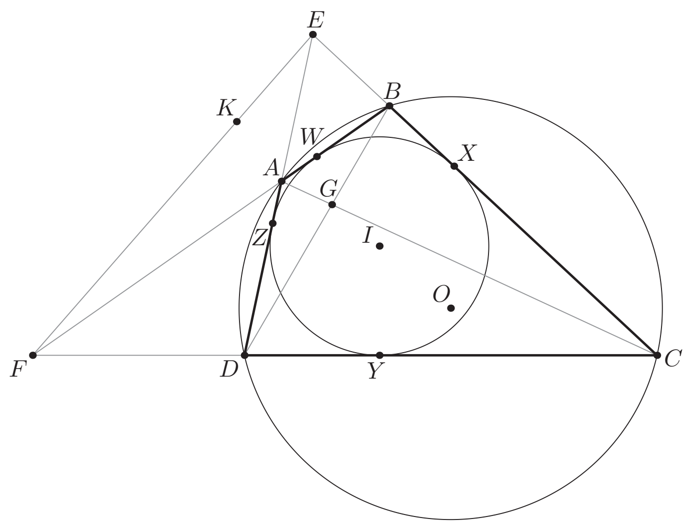

Figure 1. The bicentric quadrilateral configuration.

- $E, F,$ and G are the intersections of $\overline{AD}$ and $\overline{BC}$, $\overline{AB}$ and $\overline{CD}$, and $\overline{AC}$ and $\overline{BD}$, respectively.
- I and O are the incenter and circumcenter of $ABCD$, respectively.
- K is the Miquel Point of $ABCD$. (The *Miquel Point* is the common intersection of the circumcircles of $\triangle ECD, \triangle FBC, \triangle EBA,$ and $\triangle FAD$.)
- $W, X, Y,$ and Z are the points at which the incircle meets sides $\overline{AB}, \overline{BC}, \overline{CD},$ and $\overline{DA}$, respectively.
- ω and Ω are the incircle and circumcircle of $ABCD$, respectively.

Publication Date: February 4, 2019. Communicating Editor: Paul Yiu.

Next, we present a number of well-known facts about this configuration, which will be used throughout the rest of the paper.

Proposition 1. *In bicentric quadrilateral $ABCD$,*

(a) W, G, Y *and* Z, G, X *are collinear, with* $\overline{WGY} \perp \overline{ZGX}$.

(b) *Points* O, G, K *are collinear with* $\overline{OGK} \perp \overline{EF}$.

(c) *Points* I, G, K *are collinear; hence,* O, I, G, *and* K *are collinear, and* $\overline{OIGK} \perp \overline{EF}$.

(d) $\angle EIF$ *is a right angle.*

Proof. (a) The collinearities follow from applications of Brianchon's Theorem on the two degenerate hexagons $AWBCYD$ and $ABXCDZ$, which gives that G is also the intersection of $\overline{WGY}$ and $\overline{ZGX}$. That $\overline{WGY} \perp \overline{ZGX}$ has been shown in [2].

(b) In fact, K is the inverse of G with respect to Ω ([3], Theorem 10.12), so O, G, K collinear is immediate. Meanwhile, by Brokard's Theorem, $\overline{EF}$ is the polar of G with respect to Ω, immediately implying $\overline{OGK} \perp \overline{EF}$.

(c) It has been shown that $\frac{AZ}{ZD} = \frac{BX}{XC}$ in [4]; hence, as K is the center of the spiral similarity mapping $\overline{AB}$ to $\overline{DC}$, it also maps $\overline{AB}$ to $\overline{ZX}$. Thus, by [6], it follows that $EKZX$ is a cyclic quadrilateral; furthermore, as $\angle IZE = \angle IXE = 90^\circ$, we know that E, K, Z, I, and X are concyclic.

In a similar manner we may show that F, K, W, I, and Y are concyclic. Now, inversion about ω maps the circumcircle of $WIYFK$ to $\overline{WY}$ and the circumcircle of $XIZKE$ to $\overline{XZ}$; thus K maps to G, and the fact is proven.

(d) This follows immediately from $\overline{EI} \perp \overline{ZX}$ and $\overline{FI} \perp \overline{WY}$ and see (a). $\square$

2. Radical Axis

We will show the following result:

Theorem 2. *The radical axis of* Ω *and* ω *is the* G*-midline in* $\triangle EFG$.

Let Γ_E be the E-mixtilinear incircle of $\triangle EDC$—that is, the circle tangent to sides $\overline{ED}$ and $\overline{EC}$, and internally tangent to the circumcircle of $\triangle EDC$.

Theorem 3. *The radical center of* Γ_E, ω, *and* Ω *is the midpoint of* $\overline{FG}$, *point* P.

Proof. First, suppose that Γ_E meets $\overline{ED}$ and $\overline{EC}$ at Z_1 and X_1, respectively; it is well-known ([1], [5]) that I lies on $\overline{Z_1X_1}$.

Now, as $\overline{ZZ_1}$ and $\overline{XX_1}$ are the common external tangent segments between ω and Γ_E, it follows that the radical axis of Γ_E and ω is the line passing through the midpoints of $\overline{ZZ_1}$ and $\overline{XX_1}$. Then it must follow that this radical axis passes through P.

Next, we will show that P lies on the radical axis of Ω and Γ_E. Let (EF) denote the circle with diameter $\overline{EF}$. We begin with a lemma:

Lemma 4. (EF) *is orthogonal to both* Ω *and* Γ_E.

Proof. We begin by noting that as K was defined to be the Miquel Point, $KADF$ is cyclic; thus inversion about E swaps the pairs $\{K, F\}, \{A, D\}$, and $\{B, C\}$.

Note also that EIF is a right triangle (see Proposition 1(d)) with K as the foot of the I-altitude, so $EK \cdot EF = EI^2$; thus I maps to itself under this inversion.

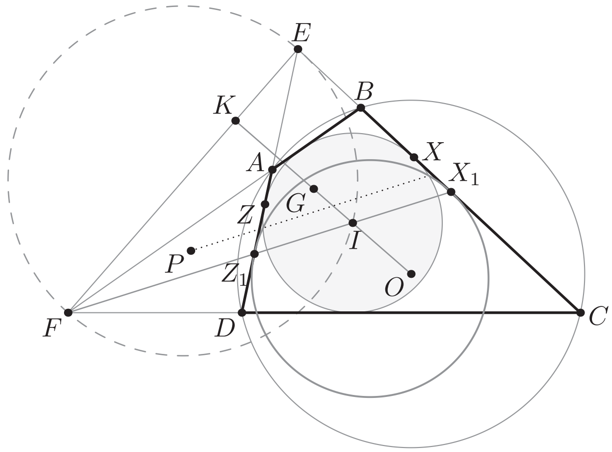

Figure 2. P lies on the radical axis of Γ_E and ω. (EF) is orthogonal to Ω and Γ_E.

As EIZ_1 is a right triangle with Z as the foot of the I-altitude, it follows that the same inversion swaps $\{Z, Z_1\}$, and similarly that it swaps $\{X, X_1\}$. Thus Γ_E swaps with ω and (EF) swaps with $\overline{OIGK}$; it follows that (EF) and Γ_E are indeed orthogonal, because $\overline{OIGK}$ passes through the center of ω.

Also, as Ω remains invariant under the inversion and $\overline{OIGK}$ passes through its center too, Ω and (EF) are also orthogonal. The lemma is proven. $\square$

Let N be the center of (EF); or, that is, the midpoint of $\overline{EF}$. It follows from our lemma that N has equal power with respect to Γ_E and Ω; thus N lies on the radical axis of Γ_E and Ω.

We will now prove one more lemma:

Lemma 5. *If O_E is the center of Γ_E, then O, O_E, and F are collinear.*

Proof. We will show, equivalently, that $\overline{EG}$ is the polar of F with respect to both Γ_E and Ω.

It is obvious that $\overline{EG}$ is the polar of F with respect to Ω: that follows immediately from Brokard's Theorem applied to cyclic quadrilateral $ABCD$.

We now observe that F lies on $\overline{Z_1X_1}$ as $\overline{IF} \perp \overline{IE}$ and $\overline{Z_1IX_1} \perp \overline{EI}$. Thus E lies on the polar of F with respect to Γ_E.

To show that G lies on this polar, we will show equivalently that $(\overline{EF}, \overline{EG}; \overline{EZ_1}, \overline{EX_1})$ is a harmonic bundle. Suppose that $\overline{ZGX}$ meets $\overline{EF}$ again at J; we want to show that $(J, G; Z, X)$ is harmonic, because then we are done.

Observe that $\angle GKJ = 90^\circ$; also note that I is the midpoint of arc $\widehat{ZX}$ not containing E on the circumcircle of $EKZIX$. Thus $\overline{KG}$ bisects $\angle ZKX$; by a well-known condition ([3], Lemma 9.18), this implies that $(J, G; Z, X)$ is indeed harmonic. Thus we have shown that $\overline{EG}$ is the polar of F with respect to both Γ_E and Ω.

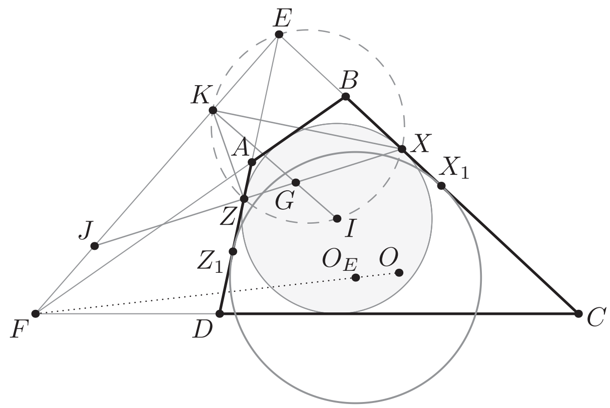

Figure 3. O, O_E and F are collinear.

Finally, this implies that $\overline{EG} \perp \overline{FO}$ and $\overline{EG} \perp \overline{FO_E}$; thus O, O_E, and F are collinear, as desired. $\square$

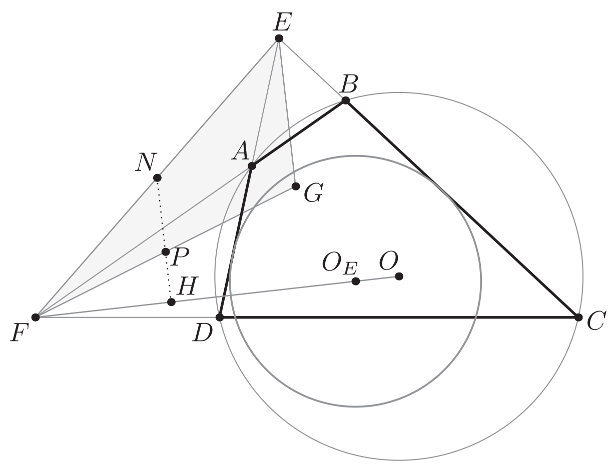

Figure 4. P lies on the radical axis of Γ_E, Ω.

We know that the radical axis of two circles is a line perpendicular to the line joining their centers. Thus, as we already know N to lie on the radical axis of Γ_E and Ω, it follows that the radical axis is the line through N perpendicular to $\overline{OO_EF}$, which is just the line through N parallel to $\overline{EG}$. This line obviously passes through point P, as $\overline{HN}$ is actually the F-midline in $\triangle EFG$.

Thus we have shown that P lies on the radical axis of Γ_E and Ω and of Γ_E and ω; it follows indeed that P is the radical center of Γ_E, ω, and Ω, as desired. $\square$

Having proven this, we may conclude a similar theorem, by symmetry. Let Γ_F be the F-mixtilinear incircle of $\triangle FBC$.

Theorem 6. *The radical center of* Γ_F, ω, *and* Ω *is the midpoint of* $\overline{EG}$, *point* Q.

Now, we are finally ready to show Theorem 2.

Proof. From Theorem 3 and Theorem 6 we see that P and Q both lie on the radical axis of Ω and ω. Thus $\overline{PQ}$—indeed the G-midline in $\triangle EFG$—is the radical axis of Ω and ω, as desired. $\square$

It is worth noting that every vertex of the medial triangle of $\triangle EFG$ is a radical center, as N is also the radical center of Γ_E, Γ_F, and Ω.

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