Forum Geometricorum
Volume 19 (2019) 45–52.

FORUM GEOM
ISSN 1534-1178

Division of an Angle into Equal Parts and Construction of Regular Polygons by Multi-Fold Origami

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Abstract. This article analyses geometric constructions by origami when up to n simultaneous folds may be done at each step. It shows that any arbitrary angle can be m-sected if the largest prime factor of m is $p \leq n+2$. Also, the regular m-gon can be constructed if the largest prime factor of $\phi(m)$ is $q \leq n+2$, where ϕ is Euler's totient function.

1. Introduction

Two classic construction problems of plane geometry are the division of an arbitrary angle into equal parts and t he construction of regular polygons [14]. It is well known that the use of straight edge and compass allows for the bisection of angles and the constructions of regular m-gons if $m = 2^a p_1 p_2 \cdots p_k$, where $a, k \geq 0$ and each p_i is a distinct odd prime of the form $p_i = 2^{b_i} + 1$. It is also known that origami extends the constructions by allowing for the trisection of angles and the constructions of regular m-gons if $m = 2^{a_1} 3^{a_2} p_1 p_2 \cdots p_k$, where $a_1, a_2, k \geq 0$ and each p_i is a distinct prime of the form $p_i = 2^{b_{i,1}} 3^{b_{i,2}} + 1 > 3$ [1].

Standard origami constructions are performed by a sequence of elementary single-fold operations, one at a time. Each elementary operation solves a set of specific incidences constraints between given points and lines and their folded images [1, 2, 8]. A total of eight elementary operations may be defined and stated as in Table 1 [12]. The operations can solve arbitrary cubic equations [3, 7], and therefore they can be applied to related construction problems such as the duplication of the cube [15] and those mentioned above [3, 4, 5].

The range of origami constructions may be extended further by using multi-fold operations, in which up to n simultaneous folds may be performed at each step [2], instead of single folds. In the case of $n = 2$, the set of possible elementary operations increases to 209 or more (the exact number has still not been determined). It has been shown that 2-fold origami allows for the geometric solution of arbitrary septic equations [9], quintisection of an angle [10] and construction of the regular hendecagon [13].

Publication Date: February 4, 2019. Communicating Editor: Paul Yiu.

Table 1. Single-fold operations [12]. $\mathcal{O}$ denotes the medium in which folds are performed; e.g., a sheet of paper, fabric, plastic, metal or any other foldable material.

#	Operation
1	Given two distinct points P and Q, fold $\mathcal{O}$ to place P onto Q.
2	Given two distinct lines r and s, fold $\mathcal{O}$ to align r and s.
3	Fold along a given a line r.
4	Given two distinct points P and Q, fold $\mathcal{O}$ along a line passing through P and Q.
5	Given a line r and a point P, fold $\mathcal{O}$ along a line passing through P to reflect r onto itself.
6	Given a line r, a point P not on r and a point Q, fold $\mathcal{O}$ along a line passing through Q to place P onto r.
7	Given two lines r and s, a point P not on r and a point Q not on s, where r and s are distinct or P and Q are distinct, fold $\mathcal{O}$ to place P onto r, and Q onto s.
8	Given two lines r and s, and a point P not on r, fold $\mathcal{O}$ to place P onto r, and to reflect s onto itself.

Thus, the purpose of this article is to analyze the general case of n-fold origami with arbitrary $n \geq 1$ and determine what angle divisions and regular polygons can be obtained.

2. Single- and multi-fold origami

An n-fold elementary operation is the resolution of a minimal set of incidence constraints between given points, lines, and their folded images, that defines a finite number of sets of n fold lines [2]. For the case of $n = 1$, all possible elementary operations are those listed in Table 1. An example of operation for $n = 2$ is illustrated in Fig. 1.

Any number of n_i-fold operations, $i = 1, 2, \ldots, k$, may be gather together and considered as a unique n-fold operation, with $n = \sum_{i=1}^{k} n_i$. Thus, we define n-fold origami as the construction tool consisting of all the k-fold elementary operations, with $1 \leq k \leq n$.

The medium on which all folds are performed is assumed to be an infinite Euclidean plane. Points are referred by their Cartesian xy-coordinates or by identifying them as complex numbers, as convenient. A point or complex number is said to be *n-fold constructible* iff it can be constructed starting from numbers 0 and 1 and applying a sequence of n-fold operations. It has been shown that the set of constructible numbers in $\mathbb{C}$ by single-fold origami is the smallest subfield of $\mathbb{C}$ that is closed under square roots, cube roots and complex conjugation [1]. An immediate corollary is that the field $\mathbb{Q}$ of rational numbers is n-fold constructible, for any $n \geq 1$.

The present analysis is based on the following version of a previous theorem by Alperin and Lang [2].

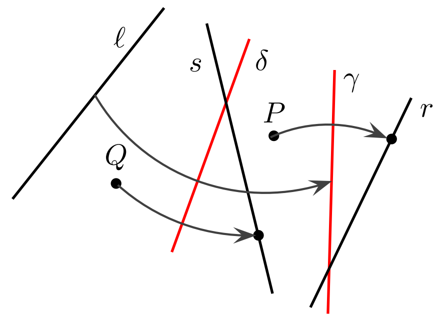

Figure 1. A two-fold operation [13]. Given two points P and Q and three lines ℓ, r, s, simultaneously fold along a line γ to place P onto r, and along a line δ to place Q onto s and to align ℓ and γ.

Theorem 1. *The real roots of any mth-degree polynomial with n-fold constructible coefficients are n-fold constructible if $m \leq n+2$.*

Proof. The real roots of any mth-degree polynomial may be obtained by Lill 's method [11, 7, 17]. It consists of defining first a right-angle path from and origin O to a terminus T, where the lengths and directions of the path's segments are given by the non-zero coefficients of the polynomial. Next, a second right-angle path with m segments between O and T is constructed by folding, and this construction demands the execution of $m-2$ simultaneous folds, if $m \geq 3$, or a single fold, if $m \leq 3$. The first intersection (from O) between both paths is the sought solution.

Details of the method may be found in the cited references. An example for solving $x^5 - a = 0$ is shown in Fig. 2. □

It must be noted that the roots of $5th$- and $7th$-degree polynomials may be obtained by 2-fold origami, instead of the 3- and 5-fold origami, respectively, predicted by the above theorem [16, 9]. Therefore, Theorem 1 only posses a sufficient condition on the number of simultaneous folds required.

3. Angle section

Let us consider first the case of division into any prime number of parts.

Lemma 2. *Any angle may be divided into p equal parts by n-fold origami if p is a prime and $p \leq n+2$.*

Proof. Let ℓ be a line forming an angle θ with the x-axis on the plane. Then, point $P(\cos\theta, 0)$ may be constructed as shown in Fig. 3.

Consider next the multiple angle identity

$$\cos(p\alpha) = T_p(\cos\alpha) \tag{1}$$

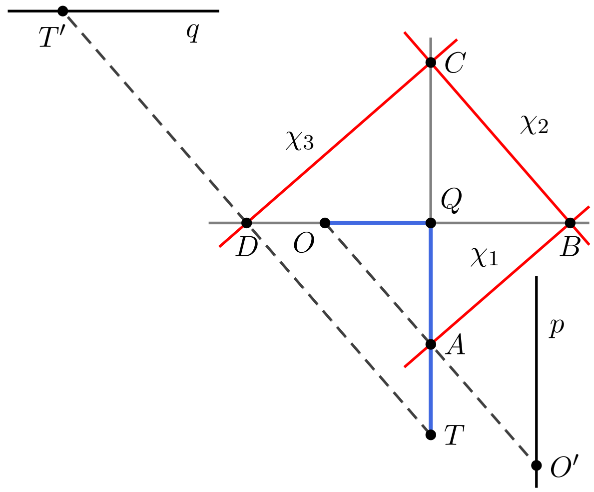

Figure 2. Geometrical solution of $x^5 - a = 0$ by 3-fold origami. Set perpendicular segments $\overline{OQ}$ and $\overline{QT}$ with respective lengths 1 and a, line p parallel to $\overline{QT}$ at a distance of 1, and line q parallel to $\overline{OQ}$ at a distance of a. Next, construct Lill's path $\overline{OA}$, $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DT}$ by performing three simultaneous folds: fold χ_1 places point O onto line p, fold χ_2 is perpendicular to χ_1 and passes through the intersection of χ_1 with the direction line of $\overline{OQ}$ (point B), and fold χ_3 is perpendicular to χ_2, passes through the intersection of χ_2 with the direction line of $\overline{QT}$ (point C), and places point T onto line q. Point A is at the intersection of χ_1 with the direction line of QT, and the length of $\overline{QA}$ is $\sqrt[5]{a}$.

where T_p is the pth Chebyshev polynomial of the first kind, defined by

$$T_0(x) = 1, \tag{2}$$

$$T_1(x) = x, \tag{3}$$

$$T_{p+1}(x) = 2xT_p(x) - T_{p-1}(x). \tag{4}$$

Letting $\theta = p\alpha$, then Eq. (1) is a pth-degree polynomial equation on $x = \cos(\theta/p)$ with integer (constructible) coefficients. According to Theorem 1, the equation may be solved by $p-2$-fold origami, if $p \geq 3$, or single-fold origami, if $p \leq 3$. Then, a line ℓ' forming an angle θ/p may be constructed from $\cos(\theta/p)$ by reversing the procedure in Fig. 3. □

The lemma is easily extended to the general case of division into an arbitrary number of parts.

Theorem 3. *Any angle may be divided into $m \geq 2$ equal parts by n-fold origami if the largest prime factor p of m satisfies $p \leq n+2$.*

Proof. Let $m = p_1 p_2 \cdots p_k$, where each p_i is a prime and $p_i \leq n+2$. Then, the theorem is proved by induction over k and applying Lemma 2. □

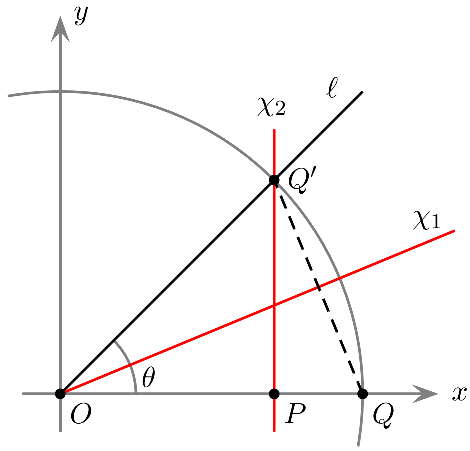

Figure 3. Construction for Lemma 2. Given points $O(0,0)$, $Q(1,0)$, and line ℓ forming an angle θ with $\overline{OQ}$: (1) fold along a line (χ_1) to place ℓ onto $\overline{OQ}$, and next (2) fold along a perpendicular (χ_2) to $\overline{OQ}$ passing through Q'. The intersection of $\overline{OQ}$ and χ_2 is $P = (\cos\theta,\, 0)$.

Again, we remark that the above theorem only posses a sufficient condition on the number of multiple folds required. For $m = 5$, it predicts $n = 3$; however, a solution using only 2-fold origami has been published [10].

Example 1. Any angle may be divided into 11 equal parts by 9-fold origami.

4. Regular polygons

The analysis follows similar steps to previous treatments on geometric constructions by single-fold origami and other tools [6, 18, 19].

Consider an m-gon ($m \geq 3$) circumscribed in a circle with radius 1 and centered at the origin in the complex plane. Its vertices are given by the mth-roots of unity, which are the solutions of $z^m - 1 = 0$.

Let us recall that an mth root of unity is primitive if it is not a kth root of unity for $k < m$. The primitive mth roots are solutions of the mth cyclotomic polynomial

$$\Phi_m(z) = \prod_{\substack{1\le k\le m\\ \gcd(k,m)=1}} \left(z - e^{2i\pi k/m}\right). \tag{5}$$

This polynomial has degree $\phi(m)$, where ϕ is Euler's totient function; i.e., $\phi(m)$ is the number of positive integers $k \leq m$ that are coprime to m. A property of any mth primitive root ξ_m is that all the m distinct roots may be obtained as ξ_m^k, for $k = 0, 1, \ldots, m-1$. This property provides a convenient way to construct the regular m-gon.

Lemma 4. *The regular m-gon is n-fold constructible if a primitive mth root of unity is n-fold constructible.*

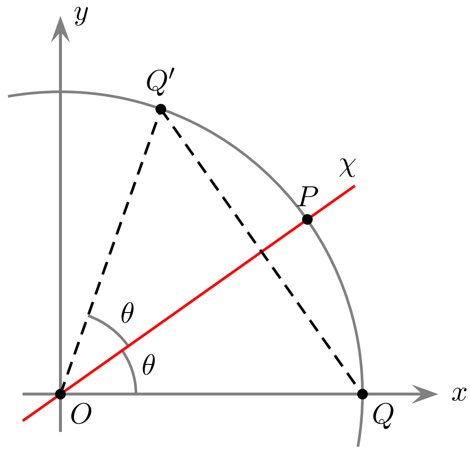

Figure 4. Given $O = (0, 0)$, $Q = (1, 0)$ and $P = (\cos\theta, \sin\theta)$, a fold along line χ passing through O and Q places Q on $Q' = (\cos 2\theta, \sin 2\theta)$.

Proof. Let $\xi_m = e^{i\theta}$ be a primitive mth root of unity. Then, $\xi_m^k = e^{ik\theta}$ and therefore all roots may be constructed from ξ_m by applying rotations of an angle θ around the origin. The rotations may be performed by single-fold origami, as shown in Fig. 4. Once all the roots have been constructed, segments connecting consecutive roots may be created by single folds. □

Next, we state a sufficient condition for the n-fold constructability of a number $\alpha \in \mathbb{C}$.

Lemma 5. *A number $\alpha \in \mathbb{C}$ is n-fold constructible if there is a field tower $\mathbb{Q} = F_0 \subseteq F1 \subseteq \cdots \subseteq F_{k-1} \subseteq F_k \subset \mathbb{C}$, such that $\alpha \in F_k$ and $[F_j : F_{j-1}] \in \{2, 3, \ldots, n+2\}$for each $j = 1, 2, \ldots, k$.*

Proof. The theorem is proved by induction over k. If $k = 0$, then $\alpha \in F_0 = \mathbb{Q}$ is constructible by single-fold origami [1], and therefore is n-fold constructible for any $n \geq 1$.

Next, assume that F_{k-1} is n-fold constructible. Let $\alpha \in F_k$, then α is a root of a minimal polynomial p with coefficients in F_{k-1}, and its degree divides $[F_k : F_{k-1}]$. If α is real, then it may be constructed by n-fold origami (Theorem 1). If not, then its complex conjugate $\bar{\alpha}$ is also a root of p. The real and imaginary parts of α, $\Re(\alpha) = (\alpha + \bar{\alpha})/2$ and $\Im(\alpha) = (\alpha - \bar{\alpha})/2$, respectively, are in F_k and therefore they are real roots of minimal polynomials $p_\Re$ and $p_\Im$ with coefficients in F_{k-1}. Again, the degrees of both $p_\Re$ and $p_\Im$ divide $[F_k : F_{k-1}]$ and hence $\Re(\alpha)$ and $\Im(\alpha)$ are n-fold origami constructible. □

Using the above lemmas, we finally obtain a sufficient condition for the constructability of the regular m-gon.

Theorem 6. *The regular m-gon is n-fold constructible if the largest prime factor p of $\phi(m)$ satisfies $p \leq n + 2$.*

Proof. Let $\phi(m) = p_1 p_2 \cdots p_k$, where each p_i is a prime and $p_i \leq n+2$, and ξ_m be a primitive mth root of unity. The Galois group Γ of the extension $\mathbb{Q}(\xi_m) : \mathbb{Q}$ is abelian and has order $\phi(m)$ [18]. Therefore, it has a series of normal subgroups $1 = \Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq \Gamma_r = \Gamma$ where each factor $\Gamma_{j+1}/|\Gamma_j$ is abelian and has order p_i for some $1 \leq i \leq k$. By the Galois correspondence, there is a field tower $\mathbb{Q}(\xi_m) = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_r = \mathbb{Q}$ such that $[K_j : K_{j+1}] = p_i$. Thus, by Lemma 5, ξ_m is n-fold constructible, and by Lemma 4, the m-gon is n-fold constructible. □

Example 2. The totient of 199 is $\phi(199) = 2 \cdot 3^2 \cdot 11$. Therefore, the regular 199-gon may be constructed by 9-fold origami.

5. Final comments

Gleason [6] noted that any regular m-gon may be constructed if, in addition to straight edge and compass, a tool to p-sect any angle is available for every prime factor p of $\phi(m)$. The above results match his conclusion: if n-fold origami can p-sect any angle for every prime factor p of $\phi(m)$, then, by Lemma 2, the largest prime factor is $p_{\max} \leq n+2$. By Theorem 6, the m-gon can be construted.

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