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Some Archimedean Circles in an Arbelos

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Abstract. We construct six circles congruent to the Archimedean twin circles in the arbelos.

For a point C on the segment AB, let AC, BC and AB be semicircles constructed on the same side. The area bounded by the three semicircles is called an arbelos. The perpendicular passing through C divides the arbelos into two curvilinear triangles with congruent incircles. Circles congruent to those circles are said to be Archimedean. Let a and b be the radii of semicircles AC and BC, respectively. The common radius of Archimedean circles is given by $\frac{ab}{a+b}$. In this paper we introduce some Archimedean circles, which we hope to be new. More examples of Archimedean circles can be found in [1] and [2].

Theorem 1. *In an arbelos, from E, draw a tangent line to semicircle AC, at F. Construct G similarly. Let H be the intersection of tangent lines EF and DG. Let circle (H, CH) cut semicircles AC and BC at I and J, respectively. Call I' the orthogonal projection of I onto AB. Construct J' similarly. Let S be the center of circle bounded by circle (A, AJ'), circle (B, BI'), and semicircle AC. Construct circle centered at T similarly. Then, circles centered at S and T are Archimedean twins (See Figure 1).*

Proof. Let CI cut DH at M. Let CJ cut EH at N. Notice that $CDIH$ is a kite, then, CI is perpendicular to DH and, as a consequence, CM is parallel to $EG = b$. As $\triangle CDM \sim \triangle EDG$, we have

$$\frac{b}{\frac{CI}{2}} = \frac{a+b}{a}$$

$$CI = \frac{2ab}{a+b}.$$

Remark. Notice that circle with diameter CI is Archimedean. This is Archimedean circle O_{50_a} in [2].

Notice also that $\triangle CII' \sim \triangle EDG$, then,

$$\frac{CI'}{b} = \frac{CI}{a+b} = \frac{\frac{2ab}{a+b}}{a+b} = \frac{2ab}{(a+b)^2}$$

$$CI' = \frac{2ab^2}{(a+b)^2}.$$

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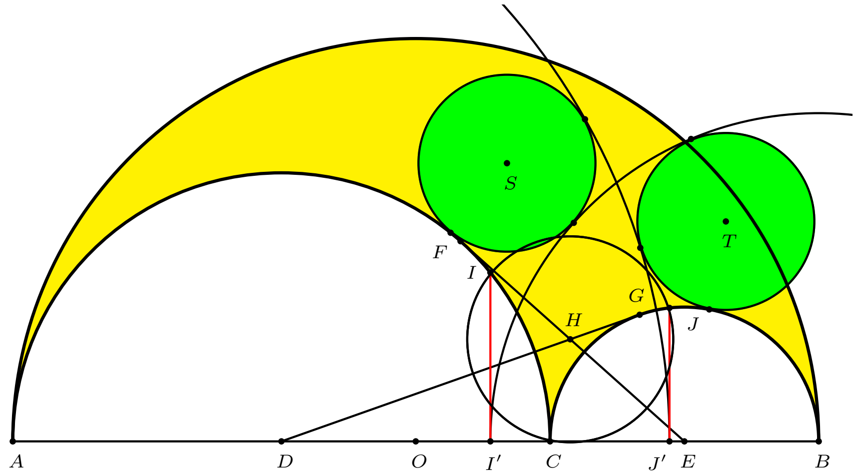

Figure 1. Circles with centers S and T are Archimedean twins

Similarly, notice that $\triangle CEN \sim \triangle DEF$ (See Figure 2), then,

$$\frac{a}{CN} = \frac{a+b}{b}$$

$$CN = \frac{ab}{a+b}.$$

Notice that $\triangle CEN \sim \triangle CJJ'$, then,

$$\frac{CJ'}{\frac{ab}{a+b}} = \frac{\frac{2ab}{a+b}}{b} = \frac{2a}{a+b}$$

$$CJ' = \frac{2a^2b}{(a+b)^2}.$$

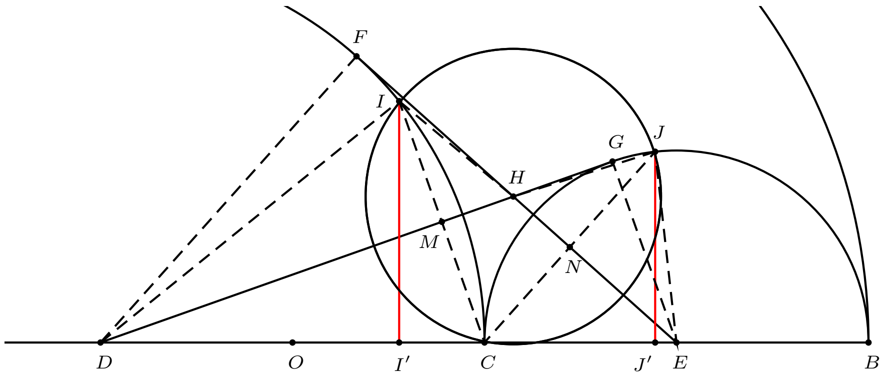

Figure 2. $\triangle CEN \sim \triangle DEF$ and $\triangle CEN \sim \triangle CJJ'$

Focusing on $\triangle ASB$ and cevian DS, if we call x the radius of circle centered at S, from the Stewart's theorem we have

$$\left[2a+\frac{2a^2b}{(a+b)^2}-x\right]^2(a+2b)+\left[2b+\frac{2ab^2}{(a+b)^2}+x\right]^2a=(2a+2b)[(a+x)^2+a(a+2b)].$$

Expanding, solving for x and simplifying we get

$$x=\frac{ab}{a+b}.$$

□

A similar reasoning goes for circle centered at T.

Theorem 2. *In an arbelos, let semicircle DE cut semicircles AC and BC in F and G, respectively. Let H be on chord DF such that $\angle CHF = 90°$. Similarly, construct I. Then, circles with radii HF and IG are Archimedean (See Figure 3).*

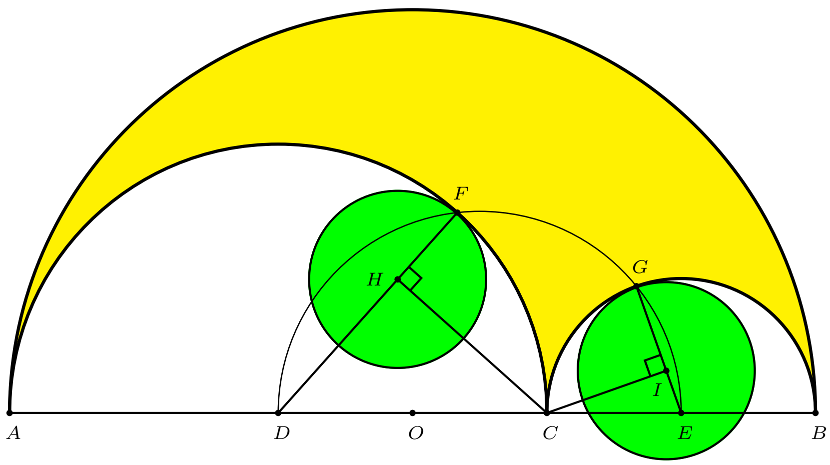

Figure 3. Circles with radii HF and IG are Archimedean twins.

Proof. Because of Thales's theorem, $DF = a$ is perpendicular to EF. As a consequence, $\triangle CDH \sim \triangle EDF$. It follows

$$\frac{a}{DH}=\frac{a+b}{a}$$

$$DH=\frac{a^2}{a+b}$$

$$FH=a-\frac{a^2}{a+b}=\frac{ab}{a+b}.$$

□

A similar reasoning must show the congruency for circle with radius GI.

Theorem 3. *In an arbelos, let the circle with center at B and radius BC meet semicircle AB in F. Similarly, construct I, on semicircle AB. Let AF intersect semicircle AC in G. Let G' be the orthogonal projection of G onto AB. The circle centered at R is bounded by semicircle AB, circle (A, AC) and the line GG'. Similarly, construct the circle centered at S. Then, the circles centered at R and S are Archimedean (See Figure 4).*

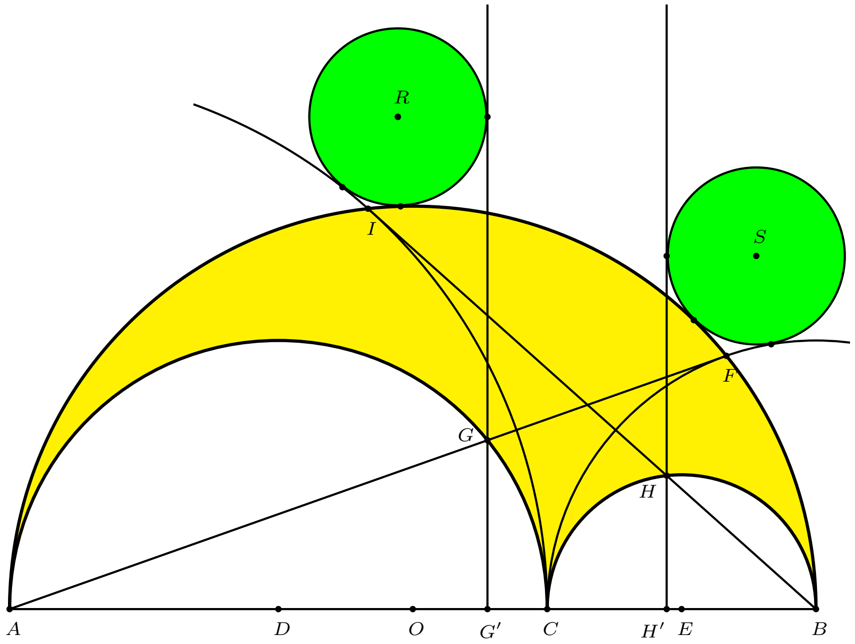

Figure 4. Circles with centers R and S are Archimedean twins.

Proof. Notice that because of Thales's theorem $\angle AGC = \angle AFB = 90^\circ$. Therefore, $\triangle AGC \sim \triangle AFB$. Thus, we have

$$\frac{GC}{2b} = \frac{2a}{2a+2b}$$

$$GC = \frac{2ab}{a+b}.$$

Remark. Notice that the circle with radius $\frac{GC}{2}$ is Archimedean.

$$AF = \sqrt{(2a+2b)^2 - 4b^2} = 2\sqrt{a^2+2ab}.$$

$$\frac{AF}{AG} = \frac{2a+2b}{2a}$$

$$AG = \frac{2a\sqrt{a^2+2ab}}{a+b}.$$

$$GC = \sqrt{4a^2 - AG^2} = \sqrt{4a^2 - \frac{4a^2(a^2+2ab)}{(a+b)^2}}$$

$$GC = \frac{2a\sqrt{4ab+b^2}}{a+b}.$$

As $\triangle AGG' \sim \triangle AGC$, it follows that

$$\frac{AG'}{AG} = \frac{AG}{AC}$$

$$AG' = \frac{2a(a^2+2ab)}{(a+b)^2}.$$

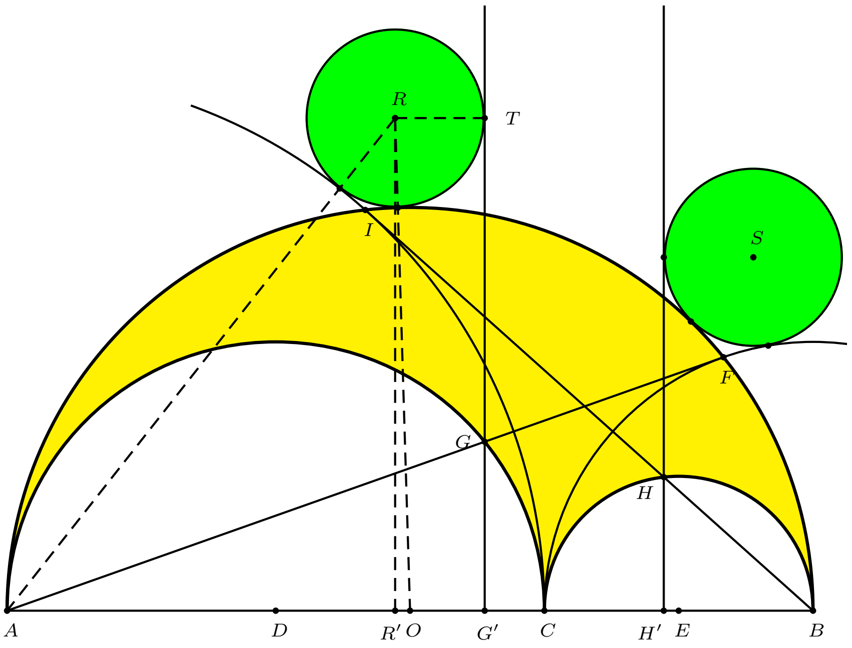

Figure 5. R' is the orthogonal projection of R onto AB.

Let R' be the orthogonal projection of R onto AB. If we call x the radius of circle centered at R, by the Pythagorean theorem (See Figure 5),

$$RR' = \sqrt{(2a+x)^2 - (AG'-x)^2}.$$

Focusing on triangle $\triangle ORR'$, by the Pythagorean theorem,

$$[AO - (AG'-x)]^2 + RR'^2 = OR^2.$$

If we replace the segments by their expressions in terms of a and b, we have the following equation

$$\left[(a+b) - \frac{2a(a^2+2ab)}{(a+b)^2} + x\right]^2 + (2a+x)^2 - \left[\frac{2a(a^2+2ab)}{(a+b)^2} - x\right]^2 = (a+b+x)^2.$$

Expanding, solving for x and simplifying we get

$$x = \frac{ab}{a+b}.$$

$\square$

A similar reasoning goes for circle centered at S.

References

[1] C. W. Dodge, T. Schoch, P. Y. Woo and P. Yiu, Those ubiquitous Archimedean circles, *Math. Mag.*, 72 (1999) 202–213.

[2] F. M. van Lamoen, Online catalogue of Archimedean circles, `http://home.kpn.nl/lamoen/wiskunde/Arbelos/Catalogue.htm`

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