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A Remark on Archimedean Incircles of an Isosceles Triangle

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Abstract. We generalize several Archimedean circles of the arbelos, which are the incircles of an isosceles triangles.

1. Introduction

We consider an arbelos configuration formed by three circles α, β and γ with diameters AO, BO and AB, respectively for a point O on the segment AB (see Figure 1). Let a and b be the radii of α and β, respectively. Circles of radius $r_{\rm A} = ab/(a+b)$ are said to be Archimedean. In [3], a special Archimedean circle is considered, which is the incircle of an isosceles triangle formed by a point lying outside of the circle γ and the two points of tangency of γ from the point. Similar Archimedean circles are also considered in [2]. In this paper we generalize those circles.

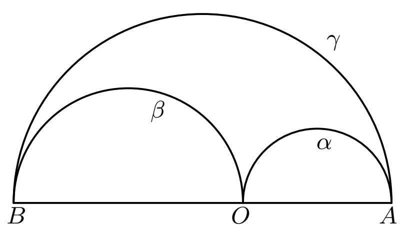

Figure 1.

2. Circles generated by a point and a circle

In this section we generalize the Archimedean circles in [2, 3].

Theorem 1. *Let δ be a circle of radius d. For a point E lying outside of δ, let F and G be the points of tangency of the tangents of δ from E and $e = |ES|$, where S is the closest point on δ to E. Then the following statements hold.*
(i) *The point S coincides with the incenter of the triangle EFG.*
(ii) *The inradius of the triangle EFG equals $de/(d+e)$.*

Proof. Assume that D is the center of δ, M is the midpoint of FS, and T is the midpoint of FG (see Figure 2). Since the triangles DMF and FTS are similar,

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$2\angle SFT = \angle FDS$, while $2\angle SFE = \angle FDS$ by the inscribed angle theorem. Hence FS bisects $\angle EFG$. Therefore S is the incenter of EFG. This proves (i). If x is the inradius of EFG, $x/e = |DF|/|DE| = d/(d+e)$. This implies $x = de/(d+e)$. □

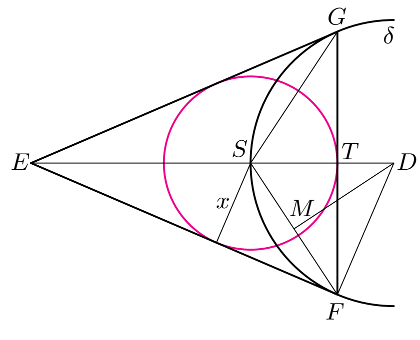

Figure 2.

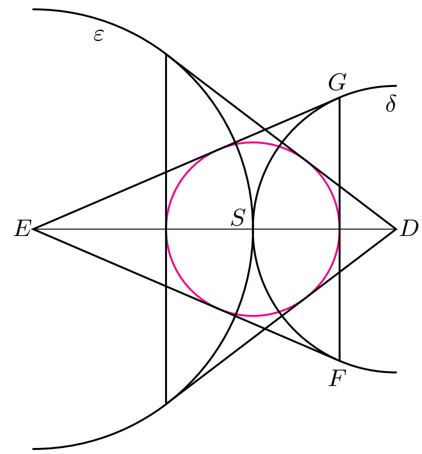

Figure 3.

Let ε be the circle of center E passing through S in Theorem 1. As the part (ii) shows, the inradius of EFG is symmetric in d and e. Hence the same circle is also constructed from the point D and the circle ε (see Figure 3).

Definition. If E is a point lying outside of a circle δ and the tangents of δ from D touches δ at points F and G, then we call the incircle of EFG the circle generated by D and δ. Also we say that the circle is generated by the circles δ and ε, where ε is the circle of center E touching δ externally.

Now we consider the arbelos. Let O_a, O_b and O_c be the centers of the circles α, β and γ, respectively.

Corollary 2. *The following statement hold.*
(i) *The circle generated by a point P and the circle α (resp. β) is Archimedean if and only if $|PO_a| = a + b$ (resp. $|PO_b| = a + b$).*
(ii) *The circle generated by a point P and the circle γ is Archimedean if and only if $|PO_c| = a + b + d$, where $d = ab(a + b)/(a^2 + ab + b^2)$.*

Proof. The part (i) is obvious by Theorem 1(ii). Solving the equation $(a+b)x/(a+b+x) = r_{\mathrm{A}}$ for x, we get $x = d$. This proves (ii). □

Notice that d is the inradius of the arbelos. The two Archimedean circles generated by a point and one of the circles α and β given in [2] are obtained in a special case in the event of Corollary 2(i). Also the Archimedean circle generated by a point and the circle γ given in [3] is obtained in a special case in the event of Corollary 2(ii). Corollary 2(i) gives an interesting special case (see Figure 4).

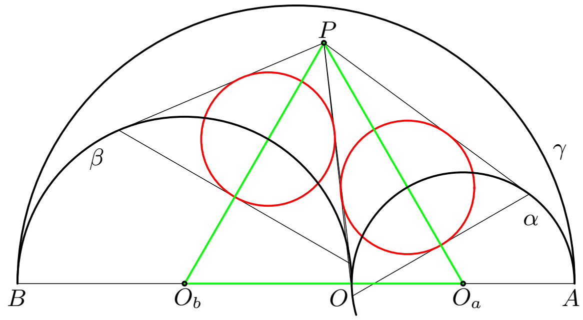

Figure 4.

Corollary 3. *If P is a point lying on the circle of center O_a (resp. O_b) congruent to γ, then the circle generated by P and α (resp. β) is Archimedean. If PO_aO_b is an equilateral triangle, then the circles generated by P and each of α and β are Archimedean.*

The Archimedean circle generated by α and β can be found in [1], which is denoted by W_8.

References

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