

Problem A3 Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x .

Solution The minimum value is $2\sqrt{2} - 1$.

Playing a little bit with the expression one discovers that if we set $\xi = \sin x + \cos x$, then

$$\sin x + \cos x + \tan x + \cot x + \sec x + \csc x = \frac{\xi^3 + \xi + 2}{\xi^2 - 1} = \xi + \frac{2}{\xi - 1}.$$

Since $\xi = \sqrt{2}\sin(x + \pi/4)$, the question can be reformulated as: Let

$$f(\xi) = \xi + \frac{2}{\xi - 1};$$

find the minimum value of $|f(\xi)|$ for $-\sqrt{2} \leq \xi \leq \sqrt{2}$. A bit of calculus shows that the minimum can only happen at $\pm\sqrt{2}$ or at a point where $f'(\xi) = 0$. There is only one such point in the interval, namely $1 - \sqrt{2}$. We see that

$$|f(-\sqrt{2})| = 3\sqrt{2} - 2, |f(1 - \sqrt{2})| = 2\sqrt{2} - 1, |f(\sqrt{2})| = 3\sqrt{2} + 2.$$

The smallest value is $2\sqrt{2} - 1$.