Solution 1 We have $\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2} = \dfrac{\left\|\overrightarrow{A_1A_2}\right\|}{\left\|\overrightarrow{BC}\right\|}\left(\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB}\right) = \vec{0}$, hence

$\overrightarrow{A_2B_1} + \overrightarrow{B_2C_1} + \overrightarrow{C_2A_1} = \vec{0}$.

So, if A' -or B', or C' - is the common point of the lines A_2B_1 and C_2A_1 - or B_2C_1 and A_2B_1, or C_2A_1 and B_2C_1 -, the triangle $A'B'C'$ is equilateral.

The triangles AC_1B_2 and BA_1C_2 are congruent, because each of them is congruent with $C'C_1C_2$; the isometry mapping AC_1B_2 to BA_1C_2 is clearly the rotation with center O and angle $\frac{2\pi}{3}$, where O is the center of the triangle ABC. Same way, the rotation $\left(O, -\frac{2\pi}{3}\right)$ maps the triangle AC_1B_2 to CB_1A_2.

Hence the triangle $A_1B_1C_1$ is equilateral with center O and the lines A_1B_2, B_1C_2 et C_1A_2 are the perpendicular bisectors of the sides of $A_1B_1C_1$; it follows that these three lines concur at O.

Solution 2 If $a_m = a_n$ with $m < n$, we have $a_m \equiv a_n\ [n+1]$ which is impossible.Hence, the a_n, for $n \in Z$, are distinct.

If $\alpha_n = \min(a_1, a_2, ..., a_n)$ and $\beta_n = \max(a_1, a_2, ..., a_n)$, as $\alpha_n \equiv \beta_n\ [\beta_n - \alpha_n]$, we have $\beta_n - \alpha_n < n$ and, $a_1, ..., a_n$ being distinct, $\{a_1, ..., a_n\}$ is a set of n consecutive integers.

As $\{a_n \mid n \in Z\}$ has infinitely many positive and negative elements, it follows that $\{a_n \mid n \in Z\} = Z$.

Solution 3 We have

$$[(x^2+y^2+z^2)x^3 - (x^5+y^2+z^2)]\left(x^2 - \frac{1}{x}\right) = \frac{(y^2+z^2)(x^3-1)^2}{x} \geq 0$$

Hence $\dfrac{x^5 - x^2}{x^5+y^2+z^2} \geq \dfrac{x^2 - \frac{1}{x}}{x^2+y^2+z^2} \geq \dfrac{x^2 - yz}{x^2+y^2+z^2}$.

As $(x^2 - yz) + (y^2 - zx) + (z^2 - xy) \geq 0$, the result follows.

Solution 4 If p is prime > 3, we have

$6.a_{p-2} = 3(2^{p-1} - 1) + 2(3^{p-1} - 1) + (6^{p-1} - 1)$ and, by Fermat's theorem, p divides a_{p-2}.

As 2 and 3 divide $a_2 = 48$, it follows that every prime p divides one of the a_n; hence 1 is the only positive integer relatively prime to all the a_n.

Solution 5 The common point Ω of the perpendicular bisectors of $[AC]$ and $[BD]$ is the center of the rotation mapping C to A, B to D and E to F. Thus Ω is the second intersection (apart D) of the circles DAP and DFQ. So, if $< d, d' >$ is the directed angle of the lines d and d', we have (modulo π)

$< RP, RQ =< AP, FQ =< AP, AD +< FD, FQ =< \Omega P, \Omega D +< \Omega D, \Omega Q =< \Omega P, \Omega Q$ and the circle PQR goes through Ω (this is also a consequence of Miquel's theorem)

Solution 6 Let $f(i,j)$ if $i \neq j$, be the number of contestants who have solved the problems i and j; let n be the number of contestants.

As $f(i,j) \geq \frac{2n+1}{5}$, wehave $S = \sum_{1 \leq i < j \leq 6} f(i,j) \geq 15\left(\frac{2n+1}{5}\right) = 6n+3$.

If each contestant has solved at most 4 problems, we have $\sum_{1 \leq i < j \leq 6} f(i,j) \leq nC_4^2 = 6n$, which is impossible.

Suppose now that only one contesant has solved 5 problems; say, the problems 1, 2, 3, 4, 5.

If p is the number of contestants who have solved exactly 4 problems, we have

$6n + 3 \leq S \leq C_5^2 + pC_4^2 + (n - p - 1)C_3^2$, hence $p \geq n - \frac{4}{3}$ and $p = n - 1$.

So $S = 10 + 6(n-1) = 6n + 4$. Moreover, $k = \frac{2n+1}{5} \in N$ because, elsewhere, $S \geq 15\left(\frac{2n+2}{5}\right) = 6n + 6$.

From $\sum_{1 \le i < j \le 6} [f(i,j) - k] = 1$, it follows that each $f(i,j)$, for $i < j$, is k except for one of them, $f(i_0,j_0)$ whose value is $k+1$.

If $\varphi(t) = \sum_{i \neq t} f(i,t)$, then $\varphi(6)$ is 3 times the number of contestants who have solved the problem 6 and, for $1 \le t < 6$, $\varphi(t) - 1$ is 3 times the number of contestants who have solved the problem t.

Hence, for $1 \le t < 6$, we have $\varphi(t) \neq \varphi(6)$.

But, if $t \notin \{i_0, j_0, 6\}$, we have $\varphi(t) = \varphi(6) = \left\{ \begin{array}{c} 5k \text{ si } j_0 < 6 \\ 5k+1 \text{ si } j_0 = 6 \end{array} \right\}$ and the contradiction.